Fractal trees for irreducible automorphisms of free groups

Thierry Coulbois

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Abstract

The self-similar structure of the attracting subshift of a primitive substitution is carried over to the limit set of the repelling tree in the boundary of Outer Space of the corresponding irreducible outer automorphism of a free group. Thus, this repelling tree is self-similar (in the sense of graph directed constructions). Its Hausdorff dimension is computed. This reveals the fractal nature of the attracting tree in the boundary of Outer Space of an irreducible outer automorphism of a free group.

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Throughout this article, F_N denotes the free group of finite rank $N \geq 2$.

An $\mathbb{R}$-tree (T, d) is an arcwise connected metric space such that two points P and Q are connected by a unique arc and this arc is isometric to the real segment $[0, d(P, Q)]$. An $\mathbb{R}$-tree is usually regarded as a 1-dimensionnal object. And, indeed, if T is a non-trivial $\mathbb{R}$-tree with a minimal action of F_N by isometries, then T is a countable union of arcs and thus has Hausdorff dimension 1.

Surprisingly, we exhibit in this article $\mathbb{R}$-trees T in the boundary of M. Culler and K. Vogtmann's Outer Space $\overline{\mathrm{CV}}_N$ (which is made of $\mathbb{R}$-trees with minimal, very-small action of F_N by isometries), such that the Hausdorff dimension of their metric completion $\overline{T}$ is strictly bigger than 1.

More precisely, we prove that, for an irreducible (with irreducible powers) outer automorphism Φ of F_N, the metric completion $\overline{T}_\Phi$ of the attracting tree T_Φ in the boundary of Outer Space has Hausdorff dimension

$$\mathrm{Hdim}(\overline{T}_\Phi) \geq \max(1; \frac{\ln \lambda_{\Phi^{-1}}}{\ln \lambda_\Phi})$$

where λ_Φ and $\lambda_{\Phi^{-1}}$ are the expansion factors of Φ and Φ^{-1} respectively. We insist that these two expansion factors may be distinct leading to a Hausdorff dimension strictly bigger than 1.

This lower bound on the Hausdroff dimension is achieved by computing the exact Hausdorff dimension of a subset of the metric completion: the limit set Ω. This is the subset of $\overline{T}_\Phi$ where the dynamic of Φ, as given by the repelling lamination concentrates.

For an irreducible (with irreducible powers) outer automorphism Φ of the free group F_N, M. Bestvina, M. Feighn and M. Handel ([BFH97]) defined the attracting lamination L_Φ. By choosing a basis $\mathcal{A}$ of F_N, the lamination L_Φ can be viewed as a symbolic dynamical system (indeed a subshift of the shift on bi-infinite reduced words in $\mathcal{A}^{\pm 1}$) as explained in [CHL08a] and briefly recalled in Section 1.1.

The attracting lamination L_Φ is best described if we choose a train-track representative $\tau = (\Gamma, *, \pi, f)$ of Φ, where Γ is a finite graph with base point $*$, π is a marking isomorphism between F_N and the fundamental group $\pi_1(\Gamma, *)$ and f is a homotopy equivalence inducing Φ via π. M. Bestvina and M. Handel ([BH92]) defined train-track representatives and proved that they always exist for irreducible (with irreducible powers) outer automorphisms of F_N (see Sections 1.3 and 1.4). The lamination L_Φ is a closed set of bi-infinite paths in the universal cover $\widetilde{\Gamma}$ of Γ, and it is invariant under application of any lift $\widetilde{f}$ of f to $\widetilde{\Gamma}$ (see Section 1.5).

Using the chart given by the train-track representative τ to describe the attracting lamination L_Φ we get in Proposition 1.1 a self-similar decomposition of L_Φ into finitely many cylinders. Self-similarity is here to be understood in the sense of graph directed constructions as introduced by [MW88] which is a generalisation of iterated function systems. We refer to [Edg08] for introduction and background on this topic.

The self-similar structure of the attracting lamination is wellknown to symbolic dynamists and a key tool to deal with it is the prefix-suffix automaton (see Section 1.7).

In this article we carry over this self-similar decompostion of the attracting lamination, which is partly folklore, to the limit set of the repelling tree $T_{\Phi^{-1}}$ of Φ in the boundary of Culler-Vogtman Outer Space. We refer to K. Vogtman's survey [Vog02] for background on Outer Space.

A construction of the repelling tree $T_{\Phi^{-1}}$ of Φ can be found in [GJLL98]. It is an $\mathbb{R}$-tree with a very small, minimal action of F_N by isometries with dense orbits. It comes with a contracting homothety H associated to the choice of a representative automorphism φ of the outer class Φ. The ratio of H is $\frac{1}{\lambda_{\Phi^{-1}}}$, where $\lambda_{\Phi^{-1}}$ is the expansion factor of Φ^{-1} (see Section 2.1).

From [LL03, LL03, CHL08b] (see Sections 2.2, 2.3 and 2.4), there exists a continuous map $\mathcal{Q}^2$ that maps the attracting lamination L_Φ into the metric completion $\overline{T}_{\Phi^{-1}}$ of the repelling tree $T_{\Phi^{-1}}$. The self-similar decomposition of the attracting lamination is carried over through $\mathcal{Q}^2$ to get a self-similar limit set Ω inside $\overline{T}_{\Phi^{-1}}$. Using the ratio $\frac{1}{\lambda_{\Phi^{-1}}}$ of the homothety H, we get the main result of this article:

Theorem 2.15. *Let Φ be an irreducible (with irreducible powers) outer automorphism of the free group F_N. Let $T_{\Phi^{-1}}$ be the repelling tree of Φ.*

The limit set $\Omega \subseteq \overline{T}_{\Phi^{-1}}$ has Hausdorff dimension

$$\delta_{\Phi^{-1}} = \mathrm{Hdim}(\Omega) = \frac{\ln \lambda_\Phi}{\ln \lambda_{\Phi^{-1}}}$$

where λ_Φ and $\lambda_{\Phi^{-1}}$ are the expansion factors of Φ and Φ^{-1} respectively.

Knowing $\delta_{\Phi^{-1}}$ we can use the Hausdorff measure in dimension $\delta_{\Phi^{-1}}$ to describe the correspondence between the unique ergodic probability measure carried by the attracting lamination and the metric of the $\mathbb{R}$-tree $T_{\Phi^{-1}}$.

We insist that the expansion factors of an irreducible (with irreducible powers) outer automorphism and its inverse are not equal in general. Surprisingly, this leads to compact subsets of an $\mathbb{R}$-tree which can be of Hausdorff dimension strictly bigger than 1 although an $\mathbb{R}$-tree is usually regarded as a 1-dimensional object.

During his beautiful course on the Mapping Class Group at MSRI in Fall 2007, L. Mosher mentioned that the convex core (see [Gui05]) of the product of the attracting and repelling trees of an irreducible (with irreducible powers) parageometric automorphism should be of Hausdorff dimension given by the ratio $\delta_{\Phi^{-1}}$ of the logarithms of the expansion factors of the automorphism and its inverse. This lead us to understand that the limit set of the repelling tree has the Hausdorff dimension $\delta_{\Phi^{-1}}$.

The main difficulty in proving our Theorem is to carefully study how the self-similar pieces of the limit set intersect. This involves describing the points that belong to more than one piece and proving that their prefix-suffix representations are periodic.

Finally in Section 3 we describe two classical examples and detail the shape of the limit sets and compact hearts.

In the end of this introduction we want to recall two more classical constructions which are sources of inspiration for our work.

The above picture is very different from the situation of pseudo-Anosov mapping classes which are a source of inspiration for studying outer automorphims. Indeed, a pseudo-Anosov homeomorphism φ of a hyperbolic surface and its inverse have the same expansion factor. Recall that the mapping class Φ of an homeomorphim φ of a surface S induces an outer automorphism of the fundamental group of the surface. And if the surface has non-trivial boundary, its fundamental group is a free group. The pseudo-Anosov homeomorphism φ comes with an unstable foliation $\mathcal{F}_\varphi$ on the hyperbolic surface S. Tightening this foliation we get the unstable geodesic lamination $\mathfrak{L}_\Phi$ of the mapping class Φ of which the attracting lamination of Φ is the algebraic version. Under iterations of φ, any closed curve converges to the unstable geodesic lamination.

The mapping class Φ also acts on Teichmller space and its boundary and has a repelling fixed point which can be described as an $\mathbb{R}$-tree $T_{\Phi^{-1}}$ with small action of the fundamental group of the surface. Geometrically the tree $T_{\Phi^{-1}}$ is transverse to the lift of the unstable geodesic lamination to the universal cover of the surface.

The limit set Ω of $T_{\Phi^{-1}}$ is equal to $T_{\Phi^{-1}}$ and is a countable union of intervals. Thus its Hausdorff dimension is 1 which is consistent with our Theorem.

Alternatively, following W. Thurston [FLP91], the pseudo-Anosov mapping class Φ fixes a train-track on the surface S. This train-track carries the unstable foliation. The compact sets $\Omega_{\widetilde{e}}$ (see section 2.5) for this train-track are intervals transverse to the foliation. The first return map T along the unstable foliation, on the union of these intervals is an interval exchange transformation.

Let us now review the above description in the case of an irreducible (with irreducible powers) outer automorphism Φ represented by a substitution σ. We refer to N. Pytheas Fogg [Fog02] for background and results on symbolic dynamics.

Let Φ be an outer automorphism of F_N which admits a basis $\mathcal{A}$ of F_N and a representative σ which is a substitution (that is to say, only positive letters appear in the images of the elements of $\mathcal{A}$). In this case we rather regard σ as an homomorphism of the free monoid on the alphabet $\mathcal{A}$.

Under iterations of σ, any letter $a \in \mathcal{A}$ converges to the attracting subshift Σ_σ. This is the subshift of the full shift on bi-infinite words in $\mathcal{A}$ which consists of bi-infinite words whose finite factors are factors of images of a under iterations of σ. Considering the shift map S we get a symbolic dynamic system (Σ_σ, S).

This attracting subshift Σ_σ is the (symbolic) attracting lamination L_Φ of the irreducible (with irreducible powers) outer automorphism Φ (more precisely it is half of L_Φ as we fixed, as a convention, that laminations are invariant by taking inverses). The self-similar decomposition of the attracting subshift occurs in this case in the basis $\mathcal{A}$ which is a train-track for Φ and is well-known to dynamists.

If in addition, the substitution σ satisfies the arithmetic-type Pisot condition, then the dynamical system (Σ_σ, S) has a geometric interpretation as a Rauzy fractal $\mathcal{R}_\sigma$.

The Rauzy fractal $\mathcal{R}_\sigma$ is a compact subset of $\mathbb{R}^{N-1}$. The Rauzy fractal is graphically striking when $N = 3$ in which case it is a compact subset of the plane. The Rauzy fractal comes with a piecewise exchange T. The dynamical system $(\mathcal{R}_\sigma, T)$ is semi-conjugated with the attracting subshift (Σ_σ, S).

Indeed, V. Canterini and A. Siegel [CS01] defined a map R from the attracting shif Σ_σ onto the Rauzy fractal: A bi-infinite word Z in the attracting subshift Σ_σ corresponds to the trajectory of exactly one point, $R(Z)$ of $\mathcal{R}_\sigma$. The map R is continuous and onto and therefore $\mathcal{R}_\sigma$ is a geometric representation of the dynamic of the attracting subshift.

The map R factors through the map $\mathcal{Q}^2$ which means that the Rauzy fractal is a quotient of the compact limit set $\Omega_{\mathcal{A}}$ of the repelling tree $T_{\Phi^{-1}}$ of Φ.

The self-similar decomposition of the attracting subshift Σ_σ, described by the prefix-suffix automaton, is carried over by the continuous map R to $\mathcal{R}_\sigma$. The self-similar decomposition of the Rauzy fractal $\mathcal{R}_\sigma$ obtained is the same as the self-similar decomposition of $\Omega_{\mathcal{A}}$ described in Proposition 2.5.

However, we note that the self-similar decomposition of the Rauzy fractal does not lead directly to a meaningful Hausdorff dimension because intersections between pieces may not be neglectable: the map R is non-injective in a "Hausdorff-dimension" essential way.

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If I had not been in California at that time, this paper would certainly be a joint paper with my favorite co-authors: A. Hilion and M. Lustig.

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1 Laminations and Automorphisms

1.1 Laminations

The free group F_N is Gromov-hyperbolic and has a well defined boundary at infinity ∂F_N, which is a topological space, indeed a Cantor set.

The action of F_N on its boundary is by homeomorphisms.

The **double boundary** of F_N is

$$\partial^2 F_N = (\partial F_N)^2 \smallsetminus \Delta$$

where Δ is the diagonal. An element of $\partial^2 F_N$ is a **line**.

A **lamination** (in its algebraic setting) is a closed, F_N-invariant, flip-invariant subset of $\partial^2 F_N$ (where the flip is the map exchanging the two coordinates of a line). The elements of a lamination are called **leaves**.

We refer the reader to [CHL08a] where laminations for free groups are defined and different equivalent approaches are exposed with care.

1.2 Charts and Cylinders

To give a geometric interpretation of the boundary, of leaves and of laminations we introduce charts.

Let Γ be a finite graph, with basepoint $*$ and $\pi : F_N \to \pi_1(\Gamma, *)$ a **marking** isomorphism. We say that $(\Gamma, *, \pi)$ is a **chart** for F_N.

Assigning a positive length to each edge in Γ (e.g. 1 to each edge) defines a path metric of the universal cover $\widetilde{\Gamma}$. For such a metric, $\widetilde{\Gamma}$ is a 0-hyperbolic space, indeed a tree, and it has a boundary at infinity $\partial\widetilde{\Gamma}$ which is simply the space of ends. Points of the boundary $\partial\widetilde{\Gamma}$ can be seen as infinite geodesic paths starting from a fixed lift $\widetilde{*}$ of the base point $*$.

The action of F_N on $\widetilde{\Gamma}$ by deck transformations through the marking π is by isometries. We denote by $\partial\pi : \partial F_N \to \partial\widetilde{\Gamma}$ the canonical homeomorphism between the boundaries at infinity.

Through $\partial\pi$ there is a canonical correspondence, which associates to a line $(X, Y) \in \partial^2 F_N$ the geodesic bi-infinite oriented arc of $\widetilde{\Gamma}$ $[\partial\pi(X), \partial\pi(Y)]$ joining the points at infinity $\partial\pi(X)$ and $\partial\pi(Y)$. We say that this bi-infinite geodesic path is the **geometric realisation** of the line (X, Y).

For a finite oriented geodesic arc γ in $\widetilde{\Gamma}$, the **cylinder** of γ $C_\Gamma(\gamma)$ is the set of lines whose geometric realisations contain γ.

Cylinders are closed-open sets and they form a basis of the topology of $\partial^2 F_N$. An element u of F_N translates by left multiplication the cylinder $C_\Gamma(\gamma)$ to $uC_\Gamma(\gamma) = C_\Gamma(u\gamma)$.

1.3 Automorphisms and topological representatives

Let φ be an automorphism of F_N. It extends canonically to an homeomorphism $\partial\varphi : \partial F_N \to \partial F_N$ and also induces an homeomorphism, $\partial^2\varphi$ of $\partial^2 F_N$.

For example, the inner automorphism $i_u : x \mapsto uxu^{-1}$, defined by the conjugation by the element u of F_N, acts on ∂F_N as (the left multiplication by) u.

If L is a lamination $\varphi(L) = \{(\partial\varphi(X), \partial\varphi(Y)) \mid (X, Y) \in L\}$ is also a lamination. As a lamination is invariant under the action of F_N, inner automorphisms act trivially on the set of laminations, and we get an action of the outer automorphism group $\mathrm{Out}(F_N)$ on the set of laminations. We consistently denote by $\Phi(L)=\varphi(L)$ the image of L by the outer class Φ of φ.

If $(\Gamma,*,\pi)$ is a chart for F_N as in the previous section, a **topological representative** of the outer automorphism Φ is a continuous map $f:\Gamma\to\Gamma$ which sends vertices to vertices, edges to finite reduced paths, and which is a homotopy equivalence inducing Φ through the marking π. A lift $\widetilde{f}:\widetilde{\Gamma}\to\widetilde{\Gamma}$ of f to the universal cover $\widetilde{\Gamma}$ of Γ is a **topological representative** of the automorphism $\varphi\in\Phi$ if the following condition holds:

$$\forall P\in\widetilde{\Gamma},\forall u\in F_N,\widetilde{f}(uP)=\varphi(u)\widetilde{f}(P).$$

If $\psi=i_u\circ\varphi$ is another automorphism in the outer class Φ, then $\widetilde{f}'=u\widetilde{f}$ is a topological representative of ψ: Lifts of f are in one-to-one correspondence with automorphisms in the outer class Φ.

For any lift $\widetilde{f}$ of the homotopy equivalence f of Γ, $\widetilde{f}$ is a quasi-isometry of $\widetilde{\Gamma}$ and extends to an homeomorphism, $\partial\widetilde{f}$, of the boundary at infinity $\partial\widetilde{\Gamma}$.

If $\widetilde{f}$ is a topological representative of the automorphism φ then the following diagram commutes:

$$\begin{tikzcd}
\partial F_N \arrow[r, "\partial\varphi", "\cong"'] \arrow[d, "\partial\pi", "\cong"'] & \partial F_N \arrow[d, "\partial\pi", "\cong"'] \\
\partial\widetilde{\Gamma} \arrow[r, "\partial\widetilde{f}", "\cong"'] & \partial\widetilde{\Gamma}
\end{tikzcd}$$

For a subset C of ∂^2F_N (e.g. a cylinder $C_\Gamma(\gamma)$) we abuse of notations and write: $\varphi(C)=\widetilde{f}(C)=\{(\partial\varphi(X),\partial\varphi(Y))\mid(X,Y)\in C\}$. We note that the homeomorphism $\partial^2\varphi$ maps a cylinder $C_\Gamma(\gamma)$ to a closed-open set of ∂^2F_N which is a finite union of cylinders but may fail to be a cylinder.

1.4 Train-track representatives and legal lamination

A **train-track representative** $\tau=(\Gamma,*,\pi,f)$ of the outer automorphism Φ of F_N is a chart $(\Gamma,*,\pi)$ together with a topological representative f of Φ such that for all integer $n\geq 1$, f^n is locally injective on each edge of Γ.

The lift $\widetilde{f}$ of f which is the topological representative of the automorphism $\varphi\in\Phi$ is a **train-track representative** for φ.

The train-track τ is **irreducible** if it contains no vertices of valence 1 or 2, and if Γ contains no non-trivial f-invariant subgraph.

An outer automorphism Φ is **irreducible** (with irreducible powers) if for each n, Φ^n does not fix a conjugacy class of a free factor. M. Bestvina and M. Handel [BH92] proved that irreducible (with irreducible powers) outer automorphisms always have an irreducible train-track representative.

A geodesic path γ in $\widetilde{\Gamma}$ (finite, infinite or bi-infinite) is **legal** if for all $n \geq 1$, the restriction of $\widetilde{f}^n$ to γ is injective (this does not depend on the choice of a particular lift $\widetilde{f}$ of f). In particular, from the definition, every 1-edge path is legal. A line $(X, Y) \in \partial^2 F_N$ is **legal** if its geometric realisation $[\partial\pi(X); \partial\pi(Y)]$ is a legal bi-infinite path.

The **legal lamination** L_τ of the train-track $\tau = (\Gamma, *, \pi, f)$ is the set of legal lines. From the definitions it is clear that

$$\Phi(L_\tau) \subseteq L_\tau.$$

Indeed if $\varphi \in \Phi$ is an automorphism representing the outer class Φ, $\partial^2\varphi$ sends any legal line to a legal line.

The **transition matrix** M of the homotopy equivalence f of the graph Γ is the square matrix of size the number of edges of Γ, and for each pair (e, e') of edges of Γ, the entry $m_{e,e'}$ is the number of occurences of e in the path $f(e')$. We insist that occurences are positive and are counted without taking in account orientation.

The **expansion factor** λ_Φ of the outer automorphism Φ is the Perron-Frobenius eigen-value $\lambda_\Phi > 1$ of the transition matrix M of the irreducible train-track representative $\tau = (\Gamma, *, \pi, f)$ of Φ. The expansion factor does not depend on the choice of a particular irreducible train-track representative. We denote by $(\mu_e)_e$ a positive Perron-Frobenius eigen-vector of the transition matrix M. This eigen-vector is unique up to a mulitplicative constant.

1.5 Attracting lamination

In [BFH97] the attracting lamination of an irreducible (with irreducible powers) outer automorphism is defined.

Let $\tau = (\Gamma, *, \pi, f)$ be an irreducible train-track representative of the outer automorphism Φ. Fix an edge e in the universal cover $\widetilde{\Gamma}$ of Γ. The **attracting lamination** L_Φ of Φ is the set of leaves which are limits of sequences of translates of iterated images of e:

$$L_\Phi = \{(X, Y) \in \partial^2 F_N \mid \exists \varepsilon = \pm 1, \exists u_n \in F_N, (X, Y) = \lim_{n\to\infty} u_n \widetilde{f}^n(e^\varepsilon)\}.$$

Where the sequence of paths $u_n\widetilde{f}^n(e)$ converges to the leaf (X,Y) if the sequence of startpoints converges to $\partial\pi(X)$ and the endpoints to $\partial\pi(Y)$ in the topological space $\widetilde{\Gamma}\cup\partial\widetilde{\Gamma}$.

From the definition it is clear that L_Φ is closed, F_N-invariant and flip-invariant, indeed a lamination. Moreover, as $\widetilde{f}(u_n\widetilde{f}^n(e^\varepsilon)) = \varphi(u_n)\widetilde{f}^{n+1}(e^\varepsilon)$ the attracting lamination is invariant by Φ.

This definition does not depend on the particular choice of a lift $\widetilde{f}$ of f, and as τ is irreducible this does not depend either on the choice of the edge e of $\widetilde{\Gamma}$.

As τ is a train-track representative, each path $\widetilde{f}^n(e)$ is legal and thus the attracting lamination is a sublamination of the legal lamination L_τ.

It is proven in [BFH97] that the attracting lamination does only depend on the irreducible (with irreducible powers) outer automorphism Φ and not on the choice of the train-track representative τ. It is proven there that the attracting lamination L_Φ is minimal and thus is the smallest sublamination of the legal lamination L_τ such that

$$\Phi(L_\Phi) = L_\Phi.$$

1.6 Self-similar decomposition of the attracting lamination

Although we noticed that the image of a cylinder by an automorphism is not in general a cylinder, we describe in this section the image of legal lines contained in a cylinder.

Let $(X,Y) \in \partial^2 F_N$ be a legal line for the train-track representative $\tau = (\Gamma, *, \pi, f)$ of the outer automorphism Φ of F_N. The lift $\widetilde{f}$ of f which is a train-track representative of the automorphism $\varphi \in \Phi$ restricts to an homeomorphism from the geometric realisation $[\partial\pi(X);\partial\pi(Y)]$ to its image $[\partial\pi\partial\varphi(X);\partial\pi\partial\varphi(Y)]$.

As the attracting lamination is made of legal lines, for any (legal) path γ in the universal cover $\widetilde{\Gamma}$ of Γ we have

$$\varphi(C_\Gamma(\gamma)\cap L_\Phi) \subseteq C_\Gamma(\widetilde{f}(\gamma))\cap L_\Phi.$$

For any oriented edge $\widetilde{e}$ in $\widetilde{\Gamma}$ we denote by $C_{\widetilde{e}}$ the set

$$C_{\widetilde{e}} = C_\Gamma(\widetilde{e})\cap L_\Phi$$

Proposition 1.1. *Let $\varphi \in \Phi$ be an irreducible (with irreducible powers) automorphism of F_N and Φ be its outer class. Let $\tau = (\Gamma, *, \pi, f)$ be a train-track representative for Φ and $\widetilde{f}$ a lift of f to the universal cover $\widetilde{\Gamma}$ associated to φ. For any edge $\widetilde{e}$ of $\widetilde{\Gamma}$*

$$C_{\widetilde{e}} = \biguplus_{(\widetilde{e}', \widetilde{p}, \widetilde{s})} \varphi(C_{\widetilde{e}'}),$$

the finite disjoint union is taken over triples $(\widetilde{e}', \widetilde{p}, \widetilde{s})$ such that $\widetilde{e}'$ is an edge of $\widetilde{\Gamma}$, $\widetilde{p}.\widetilde{e}.\widetilde{s}$ is a reduced path in $\widetilde{\Gamma}$ and $\widetilde{f}(\widetilde{e}') = \widetilde{p}.\widetilde{e}.\widetilde{s}$.

The above decomposition of $C_{\widetilde{e}}$ does not depend on the choice of a particular automorphism φ in the outer class Φ and of its associated lift $\widetilde{f}$ of f.

Proof. By the previous remark, any leaf in $\varphi(C_{\widetilde{e}'})$ contains the edge $\widetilde{e}$ which proves that $\varphi(C_{\widetilde{e}'}) \subseteq C_{\widetilde{e}}$.

Conversely, let (X, Y) be a leaf in $C_{\widetilde{e}}$, in particular it is a legal leaf in L_Φ and there exists a legal leaf (X', Y') in L_Φ such that $\partial\varphi(X') = X$ and $\partial\varphi(Y') = Y$. The map $\widetilde{f}$ restricts to an homeomorphism between the bi-infinite geodesic paths $[\partial\pi(X'); \partial\pi(Y')]$ and $[\partial\pi(X); \partial\pi(Y)]$, and as the former is legal and the latter contains the edge $\widetilde{e}$, there exists an edge $\widetilde{e}'$ in $[\partial\pi(X'); \partial\pi(Y')]$ such that $\widetilde{f}(\widetilde{e}') = \widetilde{p}.\widetilde{e}.\widetilde{s}$ contains the edge $\widetilde{e}$. Thus, the leaf (X, Y) is in $\varphi(C_{\widetilde{e}'})$.

We now proceed to prove that the union is a disjoint union. Let $(\widetilde{e}', \widetilde{p}, \widetilde{s})$ and $(\widetilde{e}'', \widetilde{p}', \widetilde{s}')$ be two triples such that $\widetilde{f}(\widetilde{e}') = \widetilde{p}.\widetilde{e}.\widetilde{s}$ and $\widetilde{f}(\widetilde{e}'') = \widetilde{p}'.\widetilde{e}.\widetilde{s}'$. Assume that the intersection $\varphi(C_{\widetilde{e}'}) \cap \varphi(C_{\widetilde{e}''})$ is non-empty. As $\partial^2\varphi$ is a homeomorphism the intersection $C_{\widetilde{e}'} \cap C_{\widetilde{e}''}$ is non-empty and let (X, Y) be a leaf in the intersection. As before, (X, Y) is legal and $\widetilde{f}$ restricts to a homeomorphism between the geometric realizations of (X, Y) and its image. The edge $\widetilde{e}$ is in both the images of $\widetilde{e}'$ and $\widetilde{e}''$ by this homeomorphism and thus $\widetilde{e}' = \widetilde{e}''$. It follows that the two tuples are equal and that the union in the Proposition is a disjoint union.

Finally, let $\widetilde{f}'$ be another lift of f and let $\varphi' \in \Phi$ be the automorphism associated to $\widetilde{f}'$. There exists $u \in F_N$ such that $\widetilde{f}' = u\widetilde{f}$ and $\varphi' = i_u \circ \varphi$. For any edge $\widetilde{e}'$ of $\widetilde{\Gamma}$, let $\widetilde{e}'' = {\varphi'}^{-1}(u^{-1})\widetilde{e}'$. We have

$$\widetilde{f}'(\widetilde{e}'') = \widetilde{f}'({\varphi'}^{-1}(u^{-1})\widetilde{e}') = \varphi'({\varphi'}^{-1}(u^{-1}))\widetilde{f}'(\widetilde{e}') = u^{-1}u\widetilde{f}(\widetilde{e}') = \widetilde{f}(\widetilde{e}')$$

and

$$\varphi'(C_{\widetilde{e}''}) = \psi(C_{{\varphi'}^{-1}(u^{-1})\widetilde{e}'}) = \varphi'({\varphi'}^{-1}(u^{-1})C_{\widetilde{e}'}) = u^{-1}\varphi'(C_{\widetilde{e}'}) = \varphi(C_{\widetilde{e}'}).$$

This proves that the decompositions obtained for $\widetilde{f}$ and φ and for $\widetilde{f}'$ and φ' are the same. $\square$

1.7 Prefix-suffix automaton

We now use the previous Section to define the prefix-suffix automaton for a train-track representative of an irreducible (with irreducible powers) outer automorphism of the free group. This automaton is a classical tool in the case of substitutions, see [CS01] and, indeed, working with a substitution simplifies some technicalities.

Let $\tau = (\Gamma, *, \pi, f)$ be a train-track representative of the outer automorphism Φ of the free group F_N. The **prefix-suffix automaton** of τ is the finite oriented labelled graph Σ whose vertices are edges of Γ and such that there is an edge labelled by (e', p, e, s) from e to e' if and only if the reduced path $f(e')$ is equal to the reduced path $p.e.s$, where p and s are reduced paths in Γ (the **prefix** and the **suffix** respectively). We draw this edge as

$$e \xrightarrow{p,s} e'.$$

An e**-path** σ is a (finite or infinite) reduced path in Σ starting at the vertex e. The **length** $|\sigma|$ of an e-path σ is its number of edges. We denote by Σ_e the set of finite e-paths and by $\partial\Sigma_e$ the set of infinite e-paths.

For a finite or infinite e-path σ we denote by $\sigma(n)$ its n-th vertex (which is an edge of Γ). We write $\sigma(0) = e$ and in particular $\sigma(|\sigma|)$ is the terminal vertex of σ.

We now fix a lift $\widetilde{f}$ of f which is associated to the automorphism $\varphi \in \Phi$.

Let $\widetilde{e}$ be an edge in the universal cover $\widetilde{\Gamma}$ which lies above the edge e of Γ. For an edge $e \xrightarrow{p,s} e'$ of Σ there exists a unique edge $\widetilde{e}'$ of $\widetilde{\Gamma}$ which is a lift of e' and such that $\widetilde{f}(\widetilde{e}') = \widetilde{p} \cdot \widetilde{e} \cdot \widetilde{s}$ where $\widetilde{p}$ and $\widetilde{s}$ are lifts of p and s respectively.

Let σ be an e-path and denote by $e_{n-1} \xrightarrow{p_{n-1},s_{n-1}} e_n$, its n-th edge for $1 \leq n \leq |\sigma|$ (with $e_0 = e$). By induction, for any $1 \leq n \leq |\sigma|$, there exists a unique edge $\widetilde{e}_n$ of $\widetilde{\Gamma}$ which is a lift of $e_n = \sigma(n)$ and such that $\widetilde{f}(\widetilde{e}_n) = \widetilde{p}_{n-1} \cdot \widetilde{e}_{n-1} \cdot \widetilde{s}_{n-1}$ where $\widetilde{p}_{n-1}$ and $\widetilde{s}_{n-1}$ are lifts of p_{n-1} and s_{n-1} respectively (and $\widetilde{e}_0 = \widetilde{e}$). We use the notation

$$\sigma(\widetilde{e}, \widetilde{f}, n) = \widetilde{e}_n.$$

Let φ be the automorphism in the outer class Φ associated to $\widetilde{f}$. For a finite e-path σ, we define

$$C_{\widetilde{e},\sigma} = \varphi^n(C_{\sigma(\widetilde{e},\widetilde{f},n)}).$$

We remark that this definition does not depend on the choice of the automorphism φ in the outer class Φ and of the associated lift $\widetilde{f}$ of f.

Applying recursively Propostion 1.1 we get

Proposition 1.2. *Let Φ be an irreducible (with irreducible powers) automorphism of F_N and Φ be its outer class. Let $\tau = (\Gamma, *, \pi, f)$ be a train-track representative for Φ.*

For any edge $\widetilde{e}$ of $\widetilde{\Gamma}$ and any $n \in \mathbb{N}$,

$$C_{\widetilde{e}} = \biguplus_{\sigma\in\Sigma_e, |\sigma|=n} C_{\widetilde{e},\sigma} \quad \square$$

For an infinite e-path σ we denote by $C_{\widetilde{e},\sigma}$ the compact non-empty nested intersection

$$C_{\widetilde{e},\sigma} = \bigcap_{\sigma' \text{prefix of } \sigma} C_{\widetilde{e},\sigma'}$$

and we get

Proposition 1.3. *Let Φ be an irreducible (with irreducible powers) outer automorphism of F_N and let $\tau = (\Gamma, *, \pi, f)$ be a train-track representative for Φ. Let $\widetilde{e}$ be a lift of an edge e of Γ.*

Then

$$C_{\widetilde{e}} = \biguplus_{\sigma\in\partial\Sigma_e} C_{\widetilde{e},\sigma}. \quad \square$$

We denote by $\rho_{\widetilde{e}} : C_{\widetilde{e}} \to \partial\Sigma_e$ the continuous onto map, which maps any leaf (X, Y) in $C_{\widetilde{e}}$ to the unique infinite e-path σ in $\partial\Sigma_e$ such that $(X, Y) \in C_{\widetilde{e},\sigma}$.

The infinite e-path $\rho_{\widetilde{e}}(X, Y)$ is the **prefix-suffix representation** of the leaf (X, Y) with respect to its edge $\widetilde{e}$.

Fixing a lift $\widetilde{f}$ of f and its associated automorphism $\varphi \in \Phi$, the action of φ on prefix-suffix representations is easy to describe:

Lemma 1.4. *Let $\widetilde{e}_1$ be an edge of $\widetilde{\Gamma}$. Let (X, Y) be a leaf in $C_{\widetilde{e}_1}$. Let $\widetilde{e}_0$ be an edge of $\widetilde{\Gamma}$ such that $\widetilde{f}(\widetilde{e}_1) = \widetilde{p}_0 \cdot \widetilde{e}_0 \cdot \widetilde{s}_0$ contains $\widetilde{e}_0$. Then*

$$\rho_{\widetilde{e}_0}(\partial^2\varphi(X, Y)) = (e_0 \xrightarrow{p_0,s_0} e_1) \cdot \rho_{\widetilde{e}_1}(X, Y)$$

where e_0, e_1, p_0*, and* s_0 *are the projections in* Γ *of* $\widetilde{e}_0, \widetilde{e}_1, \widetilde{p}_0$*, and* $\widetilde{s}_0$ *respectively.* □

Roughly speaking this Lemma means that the action of φ on prefix-suffix representations is by the shift map. This can be made precise in the case of subsitutions. Indeed if σ is a substitution, in the basis $\mathcal{A}$ of F_N, in the outer class Φ then the rose with N petals is a a train-track representative for Φ. The universal cover $\widetilde{\Gamma}$ is the Cayley graph of F_N and instead of the attracting lamination L_Φ we consider the attracting subshift which consists of bi-infinite words in the alphabet $\mathcal{A}$. Such a bi-infinite word Z encodes a bi-infinite indexed path in $\widetilde{\Gamma}$ that contains the origin. Thus Z belongs to one of the cylinders C_a where $a \in \mathcal{A}$ is the letter at index one in Z. Its prefix-suffix representation is computed with respect to this cylinder.

These classical conventions in the case of subsitutions make the above discussion on self-similarity of cylinders of the attracting lamination and the description of the prefix-suffix automaton much simpler.

1.8 Prefix-suffix representation of periodic leaves

In this section we continue our study of the prefix-suffix automaton.

Proposition 1.5. *Let* Φ *be an irreducible (with irreducible powers) outer automorphism of* F_N *and let* $\tau = (\Gamma, *, \pi, f)$ *be a train-track representative for* Φ*. Let* $\widetilde{e}_0$ *be a lift of an edge* e_0 *of* Γ *and let* σ *be an infinite* e_0*-path in* $\partial\Sigma_{e_0}$*.*

Then the compact set $C_{\widetilde{e}_0,\sigma}$ *is finite.*

Proof. Fix an automorphism $\varphi \in \Phi$ and an associated lift $\widetilde{f}$ of f to $\widetilde{\Gamma}$. For each n, let $\widetilde{e}_n = \sigma(\widetilde{e}_0, \widetilde{f}, n)$.

The length of the nested reduced path $\widetilde{f}^n(\widetilde{e}_n)$ goes to infinity. There are two cases:

Either both extremities of the reduced path $\widetilde{f}^n(\widetilde{e}_n)$ goes to infinity in $\partial\widetilde{\Gamma}$ and then $C_{\widetilde{e},\sigma}$ contains only one element: the limit of these paths.

Or, one of the extremities (say the initial one by symmetry) converges to a vertex $\widetilde{v}$ inside $\widetilde{\Gamma}$. Let $\widetilde{v}_n$ be the initial vertex of $\widetilde{e}_n$, for n big enough, $\widetilde{f}^n(\widetilde{v}_n) = \widetilde{v}$. As the length of the nested reduced path $\widetilde{f}^n(\widetilde{e}_n)$ goes to infinity the terminal vertices of these paths converge to a point $\partial\pi(Y) \in \partial\widetilde{\Gamma}$.

For each n, let E_n be the set of edges $\widetilde{e}'_n$ in $\widetilde{\Gamma}$ such that $\widetilde{e}'_n.\widetilde{e}_n$ is a legal reduced path in $\widetilde{\Gamma}$. The cardinality of E_n is bounded above by the number of edges of Γ. For each n, and for each leaf (X,Y) in $C_{\widetilde{e},\sigma}$, the leaf $\partial^2\varphi^{-n}(X,Y)$ belongs to $C_\Gamma(\widetilde{e}_n)$ and thus to one of the $C_\Gamma(\widetilde{e}'_n\cdot\widetilde{e}_n)$ for an $\widetilde{e}'_n$ in E_n. Thus we can write

$$C_{\widetilde{e},\sigma}\subseteq\bigcup_{\widetilde{e}'_n\in E_n}C_\Gamma(\widetilde{f}^n(\widetilde{e}'_n.\widetilde{e}_n)).$$

We can order each of the $E_n=\{\widetilde{e}^1_n,\widetilde{e}^2_n,\dots,\widetilde{e}^{r_n}_n\}$ such that the reduced finite paths $(\widetilde{f}^n(\widetilde{e}^k_n))_{n\in\mathbb{N}}$ are nested. For n big enough the terminal vertex of $\widetilde{f}^n(\widetilde{e}^k_n)$ is $\widetilde{v}$ while the lengths of these nested paths go to infinity and thus their initial vertices converge to a point $X_k\in\partial\widetilde{\Gamma}$. We get that the sequence of nested paths $(\widetilde{f}^n(\widetilde{e}^k_n.\widetilde{e}_n))_{n\in\mathbb{N}}$ converges to a leaf (X_k,Y) in $C_{\widetilde{e},\sigma}$. This proves that the cardinality of $C_{\widetilde{e},\sigma}$ is bounded above by the number of edges of Γ. □

The prefix-suffix representations of periodic leaves of the attracting lamination L_Φ have been described by Y. Jullian in his PhD thesis [Jul09] where he obtains the following result. We give here a proof because he only considers substitution automorphisms instead of general train-tracks but this is only a technical and minor improvement.

Proposition 1.6 ([Jul09]). *Let Φ be an irreducible (with irreducible powers) outer automorphism of F_N and let $\tau=(\Gamma,*,\pi,f)$ be a train-track representative for Φ. Let φ be an automorphism in the outer class Φ and let $\widetilde{f}$ be the associated lift of f to $\widetilde{\Gamma}$. Let $\widetilde{e}_0$ be a lift of an edge e_0 of Γ. Let (X,Y) be a leaf in $C_{\widetilde{e}_0}$ and let $\sigma=\rho_{\widetilde{e}_0}(X,Y)$ be its prefix-suffix representation.*

The leaf (X,Y) is periodic under the action of $\partial^2\varphi$ if and only if its prefix-suffix representation σ and the sequence $(\sigma(\widetilde{e}_0,\widetilde{f},n))_{n\in\mathbb{N}}$ are pre-periodic.

Proof. Let n be such that $\partial\varphi^n(X)=X$ and $\partial\varphi^n(Y)=Y$ then $\widetilde{f}^n$ restricts to an orientation preserving homeomorphism of the geometric realisation of the leaf (X,Y). For each edge e of Γ, the length of the path $f^k(e)$ increases to infinity with k. Thus either $\widetilde{f}^n$ fixes a vertex of the bi-infinite path $[\partial\pi(X),\partial\pi(Y)]$ or there exists a unique edge $\widetilde{e}_1$ of $[\partial\pi(X),\partial\pi(Y)]$ such that $\widetilde{e}_1$ is a non-extremal edge of $\widetilde{f}^n(\widetilde{e}_1)$. In the first case we choose for $\widetilde{e}_1$ the edge of $[\partial\pi(X),\partial\pi(Y)]$ which starts from the fixed vertex and lies in the same direction as $\widetilde{e}_0$. In both cases $\widetilde{f}^n(\widetilde{e}_1)$ contains $\widetilde{e}_1$ and there exists k_0 such that $\widetilde{f}^{nk_0}(\widetilde{e}_1)$ contains $\widetilde{e}_0$.

Let e_0 and e_1 be the images of $\widetilde{e}_0$ and $\widetilde{e}_1$ (respectively) in Γ. Let σ_0 be the finite e_0-path finishing at e_1 which corresponds to the fact that $\widetilde{e}_0$ is an edge of $\widetilde{f}^{nk_0}(\widetilde{e}_1)$. Let σ_1 be the finite e_1-loop in Σ_{e_1} which corresponds to the fact that $\widetilde{e}_1$ is an edge of $\widetilde{f}^n(\widetilde{e}_1)$. Then

$$\sigma = \rho_{\widetilde{e}_0}(X, Y) = \sigma_0 \cdot \sigma_1 \cdot \sigma_1 \cdot \sigma_1 \cdots$$

Moreover $\sigma_0(\widetilde{e}_0, \widetilde{f}, nk_0) = \widetilde{e}_1$ and $\sigma_1(\widetilde{e}_1, \widetilde{f}, n) = \widetilde{e}_1$, which proves that the sequence $(\sigma(\widetilde{e}_0, \widetilde{f}, n))_{n\in\mathbb{N}}$ is pre-periodic.

Conversely, assume that the prefix-suffix representation of the leaf (X, Y) is pre-periodic:

$$\sigma = \rho_{\widetilde{e}_0}(X, Y) = \sigma_0 \cdot \sigma_1 \cdot \sigma_1 \cdot \sigma_1 \cdots$$

where σ_0 is a finite reduced path in Σ_{e_0} finishing at e_1 and σ_1 is a finite loop in Σ_{e_1}. Assume that $\sigma_0(\widetilde{e}_0, \widetilde{f}, |\sigma_0|) = \widetilde{e}_1$ and $\sigma_1(\widetilde{e}_1, \widetilde{f}, |\sigma_1|) = \widetilde{e}_1$.

Then, applying Lemma 1.4 we get that for all n

$$\rho_{\widetilde{e}_1}(\partial^2\varphi^{|\sigma_0|}(X, Y)) = \sigma_1 \cdot \sigma_1 \cdot \sigma_1 \cdots = \rho_{\widetilde{e}_1}(\partial^2\varphi^{|\sigma_0|+n|\sigma|}(X, Y))$$

From Proposition 1.5, the set $C_{\widetilde{e}_1,\sigma'}$, with $\sigma' = \sigma_1 \cdot \sigma_1 \cdot \sigma_1 \cdots$, is finite and we get that there exists $m \neq n$ such that

$$\partial^2\varphi^{|\sigma_0|+m|\sigma_1|}(X, Y) = \partial^2\varphi^{|\sigma_0|+n|\sigma_1|}(X, Y).$$

This proves that (X, Y) is periodic under the action of $\partial^2\varphi$. $\square$

2 Repelling tree

We refer to K. Vogtmann [Vog02] for a survey and further references on Outer Space and actions of the free group on $\mathbb{R}$-trees.

2.1 Definition

Let Φ be an irreducible (with irreducible powers) outer automorphism of F_N. The action of Φ on the compactification of the projectivized Culler-Vogtman Outer Space $\overline{\mathrm{CV}}_N$ has exactly two fixed points $[T_\Phi]$ and $[T_{\Phi^{-1}}]$, one attracting and one repelling. The action of Φ on $\overline{\mathrm{CV}}_N$ has North-South dynamic (see [LL03]). The $\mathbb{R}$-trees T_Φ and $T_{\Phi^{-1}}$ have been described in [GJLL98]. The isometric actions of F_N on the $\mathbb{R}$-trees T_Φ and $T_{\Phi^{-1}}$ are both minimal, very small and with dense orbits.

We will focus in this article on the repelling fixed point $T_{\Phi^{-1}}$ of Φ. We note that (the metric of) this tree is only defined up to a multiplicative constant. But in this paper, we pick-up a particular tree $T_{\Phi^{-1}}$ in the projective class $[T_{\Phi^{-1}}]$

If we choose an automorphism φ in the outer class Φ, there exists a homothety, H on $T_{\Phi^{-1}}$ which is **associated** to φ (see [GJLL98]):

$$\forall P \in T_{\Phi^{-1}}, \forall u \in F_N, H(uP) = \varphi(u)H(P).$$

The fixed point of the homothety H may be in the metric completion $\overline{T}_{\Phi^{-1}}$ rather than in $T_{\Phi^{-1}}$, and we regard H as defined on this metric completion.

With this convention, as $T_{\Phi^{-1}}$ is the repelling tree of Φ, H is a contracting homothety of ratio

$$\lambda = \frac{1}{\lambda_{\Phi^{-1}}} < 1$$

where $\lambda_{\Phi^{-1}}$ is the expansion factor of the irreducible (with irreducible powers) outer automorphism Φ^{-1}.

2.2 The map $\mathcal{Q}$

Under the hypothesis of the previous section, $T_{\Phi^{-1}}$ is an $\mathbb{R}$-tree with a minimal, very small action of F_N by isometries with dense orbits. We denote by $\widehat{T}_{\Phi^{-1}} = \overline{T}_{\Phi^{-1}} \cup \partial T_{\Phi^{-1}}$ the union of its metric completion and of its Gromov boundary. The space $\widehat{T}_{\Phi^{-1}}$ comes with the topology induced by the metric on $T_{\Phi^{-1}}$. However, we consider the weaker **observers' topology** on $\widehat{T}_{\Phi^{-1}}$. We refer to [CHL07] for details on this topology. A basis of open sets is given by the **directions**: a direction is a connected component of $\widehat{T}_{\Phi^{-1}} \smallsetminus \{P\}$ where P is any point of $\widehat{T}_{\Phi^{-1}}$. We denote by $\widehat{T}^{\mathrm{obs}}_{\Phi^{-1}}$ the set $\widehat{T}_{\Phi^{-1}}$ equipped with the observers' topology. The space $\widehat{T}^{\mathrm{obs}}_{\Phi^{-1}}$ is Hausdorff, compact and has the same connected components than $\widehat{T}_{\Phi^{-1}}$. Indeed it is a dendrite in B. Bowditch [Bow99] terminology.

Theorem 2.1 ([CHL07]). *For any point $P \in \overline{T}_{\Phi^{-1}}$, the map $\mathcal{Q}_P : F_N \to \widehat{T}^{obs}_{\Phi^{-1}}, u \mapsto uP$ has a unique equivariant continuous extension to a map $\mathcal{Q} : \partial F_N \to \widehat{T}^{obs}_{\Phi^{-1}}$. This extension is independent of the choice of the point P.*

The map $\mathcal{Q}$ was first introduced in [LL03, LL08] with a slightly different approach.

Note that the map $\mathcal{Q}$ fails to be continuous if we replace the observers' topology by the stronger metric topology.

Let P be a point in $\overline{T}_{\Phi^{-1}}$ and let X be in ∂F_N. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of elements of F_N such that u_n converges to X. For each n, $H(u_nP) = \varphi(u_n)H(P)$. From Theorem 2.1, and for the observers' topology u_nP converge towards $\mathcal{Q}(X)$ while $\varphi(u_n)H(P)$ converge towards $\mathcal{Q}(\partial\varphi(X))$. Thus we have proved

Lemma 2.2. *For any element $X \in \partial F_N$, $\mathcal{Q}(\partial\varphi(X)) = H(\mathcal{Q}(X))$.* □

2.3 Dual lamination and the map $\mathcal{Q}^2$

Using the map $\mathcal{Q}$, in [CHL08b], a lamination $L(T_{\Phi^{-1}})$ dual to the tree $T_{\Phi^{-1}}$ was defined.

$$L(T_{\Phi^{-1}}) = \{(X,Y) \in \partial^2 F_N \mid \mathcal{Q}(X) = \mathcal{Q}(Y)\}.$$

From this definition, the map $\mathcal{Q}$ naturally induces an equivariant map $\mathcal{Q}^2 : L(T_{\Phi^{-1}}) \to \widehat{T}_{\Phi^{-1}}$. It is proven in [CHL08b] that the map $\mathcal{Q}^2$ is continuous (for the metric topology on $\widehat{T}_{\Phi^{-1}}$). The image Ω of $\mathcal{Q}^2$ is the **limit set** of $T_{\Phi^{-1}}$. It is contained in $\overline{T}_{\Phi^{-1}}$ (equivalently, points of the boundary $\partial T_{\Phi^{-1}}$ have exactly one pre-image by $\mathcal{Q}$) but Ω may be strictly smaller than $\overline{T}_{\Phi^{-1}}$, in particular it may fail to be connected.

From Lemma 2.2, the dual lamination $L(T_{\Phi^{-1}})$ is invariant by Φ and we deduce

Lemma 2.3. *For any leaf (X,Y) of the dual lamination $L(T_{\Phi^{-1}})$, we have $\mathcal{Q}^2(\partial^2\varphi(X,Y)) = H(\mathcal{Q}^2(X,Y))$.* □

2.4 Dual and attracting laminations

The dual lamination is sometime called the zero-length lamination and it is clear to the experts that it contains the attracting lamination. This is for example proven in [HM06].

Proposition 2.4. *The attracting lamination L_Φ of an irreducible (with irreducible powers) outer automorphism Φ is a sublamination of the lamination $L(T_{\Phi^{-1}})$ dual to the repelling tree $T_{\Phi^{-1}}$ of Φ:*

$$L_\Phi \subset L(T_{\Phi^{-1}}).$$

Proof. Let $\tau = (\Gamma, *, \pi, f)$ be a train-track representative of Φ. Let $\widetilde{f}$ be a lift of f which is associated to the automorphism φ in the outer class Φ.

Let (X, Y) be a leaf in L_Φ. Then by definition there exists an edge e of the universal cover $\widetilde{\Gamma}$ of Γ and a sequence u_n of elements of F_N such that $u_n\widetilde{f}^n(e)$ converges to (X, Y). Fix two base points in $\widetilde{\Gamma}$ and $T_{\Phi^{-1}}$ (both denoted by $*$) and consider an equivariant map $q : \widetilde{\Gamma} \to T_{\Phi^{-1}}$ such that $q(*) = *$ and which is affine on edges of $\widetilde{\Gamma}$. Then for any vertex P of $\widetilde{\Gamma}$,

$$q(\widetilde{f}(P)) = H(P).$$

We deduce that the length of $q(u_n\widetilde{f}^n(e))$ is λ^n times the length of $q(e)$ and as $\lambda < 1$ this length converges to 0 when n goes to infinity.

Let now P_0 be the start-point of e and P_1 be its end-point. Then $u_n\widetilde{f}^n(P_0)$ converges to $\partial\pi(X)$ and $u_n\widetilde{f}^n(P_1)$ converges to $\partial\pi(Y)$. The map $\mathcal{Q}$ is continuous for the weaker observers' topology on $\widehat{T}$ (see [CHL07]), so that for this observers' topology $q(u_n\widetilde{f}^n(P_0))$ converges to $\mathcal{Q}(X)$ and $q(u_n\widetilde{f}^n(P_1))$ converges to $\mathcal{Q}(Y)$. The distance $d(q(u_n\widetilde{f}^n(P_0)), q(u_n\widetilde{f}^n(P_1)))$ converges to 0. The metric topology is stronger than the observers' topology, thus the sequence $q(u_n\widetilde{f}^n(P_1))$ converges to $\mathcal{Q}(X)$. As the observers' topology is Hausdorff we conclude that $\mathcal{Q}(X) = \mathcal{Q}(Y)$. This proves that the leaf (X, Y) is in the dual lamination $L(T_{\Phi^{-1}})$ of $T_{\Phi^{-1}}$. □

2.5 Self-similar structure

Let Φ be an irreducible (with irreducible powers) outer automorphism of F_N. Let $\tau = (\Gamma, *, \pi, f)$ be an irreducible train track representative for Φ. Let $\varphi \in \Phi$ be an automorphism in the outer class Φ and $\widetilde{f}$ be the corresponding lift of f to the universal cover $\widetilde{\Gamma}$ of Γ.

Recall from Section 1.5 that for an edge $\widetilde{e}$ of $\widetilde{\Gamma}$ we denote by $C_{\widetilde{e}}$ the set of lines:

$$C_{\widetilde{e}} = C_\Gamma(\widetilde{e}) \cap L_\Phi.$$

Using the map $\mathcal{Q}^2$ of Section 2.3 and Proposition 2.4, we denote by $\Omega_{\widetilde{e}}$ the subset of $\overline{T}_{\Phi^{-1}}$:

$$\Omega_{\widetilde{e}} = \mathcal{Q}^2(C_{\widetilde{e}}) = \mathcal{Q}^2(C_\Gamma(\widetilde{e}) \cap L_\Phi).$$

As $\mathcal{Q}^2$ is continuous, $\Omega_{\widetilde{e}}$ is compact.

Using the irreducibility of the train-track τ, eadge leaf of the attracting lamination contains a translate of the edge $\widetilde{e}$, thus $L_\Phi = F_N.C_{\widetilde{e}}$ and $\Omega = F_N.\Omega_{\widetilde{e}}$.

Of course, the map $\mathcal{Q}^2$ is invariant by the flip map. If $\widetilde{e}'$ is the reversed edge of $\widetilde{e}$, the cylinders $C_{\widetilde{e}}$ and $C_{\widetilde{e}'}$ are homeomorphic by the flip map, and the correponding sets of $\overline{T}_{\Phi^{-1}}$ are equal: $\Omega_{\widetilde{e}} = \Omega_{\widetilde{e}'}$.

We now apply $\mathcal{Q}^2$ to Proposition 1.1.

Proposition 2.5. *Let $\varphi \in \Phi$ be an irreducible (with irreducible powers) automorphism of F_N and Φ be its outer class. Let $\tau = (\Gamma, *, \pi, f)$ be a train-track representative for Φ and $\widetilde{f}$ a lift of f to the universal cover $\widetilde{\Gamma}$ associated to φ.*

For each edge $\widetilde{e}$ which is a lift of the edge e of Γ

$$\Omega_{\widetilde{e}} = \bigcup_{(\widetilde{e}',\widetilde{p},\widetilde{s})} H(\Omega_{\widetilde{e}'})$$

where the finite union is taken over all triples $(\widetilde{e}', \widetilde{p}, \widetilde{s})$ such that $\widetilde{e}'$ is an edge of $\widetilde{\Gamma}$, $\widetilde{p}.\widetilde{e}.\widetilde{s}$ is a reduced path in $\widetilde{\Gamma}$ and $\widetilde{f}(\widetilde{e}') = \widetilde{p}.\widetilde{e}.\widetilde{s}$.

The above decomposition of $\Omega_{\widetilde{e}}$ does not depend on the choice of a particular automorphism φ in the outer class Φ, of the associated lift $\widetilde{f}$ of f and of the associated homothety H.

Proof. The equality follows directly by applying $\mathcal{Q}^2$ to Proposition 1.1:

$$\begin{aligned}
\Omega_{\widetilde{e}} &= \mathcal{Q}^2(C_{\widetilde{e}}) \\
&= \mathcal{Q}^2(\uplus\varphi(C_{\widetilde{e}'})) \\
&= \cup\mathcal{Q}^2(\varphi(C_{\widetilde{e}'})) \\
&= \cup H(\mathcal{Q}^2(C_{\widetilde{e}'})) \\
&= \cup H(\Omega_{\widetilde{e}'}) \quad \square
\end{aligned}$$

This self-similar structure of the compact subsets $\Omega_{\widetilde{e}}$ takes place in the metric space $T_{\Phi^{-1}}$. Thus, this is exactly that of a directed graph construction (see [MW88]) with similarity ratios equal to the ratio $\frac{1}{\lambda_{\Phi^{-1}}}$ of the homothety H.

But we lose the disjointness of the pieces in the self-similar decomposition. Indeed, the $\Omega_{\widetilde{e}'}$ involved in the decomposion may fail to be disjoint. We will address this key issue for the computation of the Hausdorff dimension in section 2.8.

2.6 The maps $\mathcal{Q}_{\widetilde{e}}$

Exactly as for cylinders of the attracting lamination, we can iterate the decomposition. Let $\widetilde{e}$ be an edge of $\widetilde{\Gamma}$ that is a lift of the edge e of Γ. For any e-path σ in $\Sigma_e \cup \partial\Sigma_e$ we consider

$$\Omega_{\widetilde{e},\sigma} = \mathcal{Q}^2(C_{\widetilde{e},\sigma}).$$

As above, using Propositions 1.2 and 1.3 we get:

Proposition 2.6. *Let Φ be an irreducible (with irreducible powers) automorphism of F_N and Φ be its outer class. Let $\tau = (\Gamma, *, \pi, f)$ be a train-track representative for Φ. Let $\widetilde{e}$ be an edge of $\widetilde{\Gamma}$. Let φ be an automorphism in the outer class Φ and let $\widetilde{f}$ be the associated lift of f. Let H be the associated homothety of the attracting tree $T_{\Phi^{-1}}$ of Φ.*

1. *For any e-path σ of length n, $\Omega_{\widetilde{e},\sigma} = H^n(\Omega_{\sigma(\widetilde{e},\widetilde{f},n)})$*
2. $\forall n \in \mathbb{N} \cup \{\infty\}, \quad \Omega_{\widetilde{e}} = \bigcup_{\sigma \in \Sigma_e, |\sigma| = n} \Omega_{\widetilde{e},\sigma}$
3. *The map $\mathcal{Q}^2$ factors through the map $\rho_{\widetilde{e}}$: there exists a continuous map $\mathcal{Q}_{\widetilde{e}} : \partial\Sigma_e \to \Omega_{\widetilde{e}}$ that makes the following diagram commutes:*

$$\begin{tikzcd} C_{\widetilde{e}} \arrow[rr, "\rho_{\widetilde{e}}"] \arrow[dr, "\mathcal{Q}^2"'] & & \partial\Sigma_e \arrow[dl, "\mathcal{Q}_{\widetilde{e}}"] \\ & \Omega_{\widetilde{e}} & \end{tikzcd} \quad \square$$

In the purpose of describing the self-similar structure of $\Omega_{\widetilde{e}}$ the choice of an orientation of each edge of $\widetilde{\Gamma}$ is irrelevant as the map $\mathcal{Q}^2$ is flip-invariant. Thus we could consider the smaller **unoriented prefix-suffix automaton** Σ^u which is obtained from the prefix-suffix automaton Σ by identifying two vertices e_1 and e_2 if they are the same edge of Γ with reverse orientations and by identifying to edges $e_1 \xrightarrow{p_1,s_1} e_1'$ and $e_2 \xrightarrow{p_2,s_2} e_2'$ if the edges e_1, e_2 and e_1' ,e_2' are the same edge of Γ with reverse orientation and if p_1, s_2 and s_1, p_2 are the same paths in Γ with reverse orientations.

In the classical context of substitutions the prefix-suffix automaton has two symmetric connected components (one with positive letters and one with inverses) and only the first one is usually considered.

2.7 Attracting current

A **current** for the free group F_N is a Radon measure (recall that a Radon measure is a Borel measure which is finite on compact sets) on the double boundary $\partial^2 F_N$ that is F_N-invariant and flip-invariant.

As currents are F_N-invariant the action of the automorphism group factors modulo inner automorphisms to a get an action of the outer automorphism group $\mathrm{Out}(F_N)$ on the space of currents.

The irreducible (with irreducible powers) outer automorphism Φ has an attracting projectivized current $[\mu_\Phi]$ which was introduced by R. Martin [Mar95]. Exactly as for the attracting tree (and the repelling tree) we pick one current μ_Φ in this projetivized class.

This current satisfies

$$\Phi.\mu_\Phi = \lambda_\Phi \mu_\Phi$$

where λ_Φ is the expansion factor of Φ. That is to say, for every measurable set $A \subseteq \partial^2 F_N$,

$$(\Phi.\mu_\Phi)(A) = \mu_\Phi(\varphi^{-1}(A)) = \lambda_\Phi \mu_\Phi(A)$$

where φ is any automorphism in the outer class Φ.

We refer to I. Kapovich [Kap06] for background, definitions and statements on currents.

R. Martin [Mar95] proved that the support of μ_Φ is exactly the attracting lamination L_Φ of Φ. It is proven in [CHL08c] that the lamination $L(T_{\Phi^{-1}})$, and thus its sublamination L_Φ is uniquely ergodic.

This (projectivized) attracting current is better described if we use the prefix suffix-automaton. Let $\tau = (\Gamma, *, \pi, f)$ be a train-track representative of Φ. Recall from Section 1.4 that we denote by $(\mu_e)_e$ a Perron-Frobenius eigen-vector of the transition matrix of τ. From the definition of $C_{\widetilde{e},\sigma}$ we get

Lemma 2.7. *For any edge $\widetilde{e}$ of $\widetilde{\Gamma}$ that lies above the edge e of Γ*

$$\mu_\Phi(C_{\widetilde{e}}) = \mu_e.$$

Let σ be a finite e-path in Σ_e that ends at the edge e' of Γ. Then

$$\mu_\Phi(C_{\widetilde{e},\sigma}) = \frac{\mu_{e'}}{(\lambda_\Phi)^{|\sigma|}}. \quad \square$$

(There is some fuzzyness in these equalities as both the eigen-vector and the attracting current are only defined up to a mulitplicative constant. The

Lemma has to be understood as: there is a choice of μ_Φ and of $(\mu_e)_e$ such that...).

We consider ν_Φ the push-forward of the attracting current μ_Φ by the continuous map $\mathcal{Q}^2$ to the repelling tree $T_{\Phi^{-1}}$: That is to say for any measurable set A in $\overline{T}_{\Phi^{-1}}$

$$\nu_\Phi(A) = \mu_\Phi({\mathcal{Q}^2}^{-1}(A)).$$

From Lemma 2.7 we get

Lemma 2.8. *For any edge $\widetilde{e}$ of $\widetilde{\Gamma}$ that lies above the edge e of Γ*

$$\nu_\Phi(\Omega_{\widetilde{e}}) = 2\mu_e.$$

Let σ be a finite e-path in Σ_e that ends at the edge e' of Γ. Then

$$\nu_\Phi(\Omega_{\widetilde{e},\sigma}) = 2\frac{\mu_{e'}}{(\lambda_\Phi)^{|\sigma|}}. \quad \square$$

The 2 factor comes from the fact that we considered currents as being invariant by the flip-map and that $\mathcal{Q}^2$ is flip-invariant. As both the attracting current and the metric of the repelling tree are only defined up to a multiplicative constant this is totally unsignificant.

2.8 (Non-)Injectivity of $\mathcal{Q}$

To get the Hausdorff dimension and measure of a self-similar metric space a key feature is to know how much the self-similar pieces are disjoint. In this purpose we collect results on the (non-)injectivity of $\mathcal{Q}$, $\mathcal{Q}^2$ and $\mathcal{Q}_{\widetilde{e}}$ and we complete Proposition 2.5 by stating that the pieces in the self-similar decomposition intersect in at most finitely many points.

Those results are much easier to state and prove in the case of non-geometric outer automorphisms of the free group. Recall that an outer automorphism Φ of the free group is **geometric** if it is induced by a homeomorphism h of a surface S with boundary such that $\pi_1(S) = F_N$. In this case h fixes the boundary components of the fundamental group of S and the action of F_N of the repelling and attracting trees $T_{\Phi^{-1}}$ and T_Φ are not free. In this geometric case we have to deal with stabilizers of points and fixed subgroups of the automorphisms in the outer class Φ. However the two trees $T_{\Phi^{-1}}$ and T_Φ are **surface** (they are transverse to the lifts of the stable and unstable foliations of h on S), their limit sets $\Omega_{\widetilde{e}}$ are intervals (or multi-interval) and the Hausdorff dimensions are 1 which is not really striking.

On the opposite, if we assume that Φ is non-geometric then the action of F_N on the repelling and attracting trees are free and automorphisms in the outer class Φ have trivial fixed subgroups. This simplifies our work. Thus from now on we assume that Φ is non-geometric.

The following result is proven in [CH08].

Proposition 2.9 ([CH08])**.** *Let Φ be an irreducible (with irreducible powers) non-geometric outer automorphism of F_N. Let $T_{\Phi^{-1}}$ be its repelling tree in the boundary of outer space.*

Then $\mathcal{Q}$ is finite-to-one and there are finitely many orbits of points in $\widehat{T}_{\Phi^{-1}}$ with strictly more than two pre-images by $\mathcal{Q}$. $\square$

From the definitions of $\mathcal{Q}^2$ and, if we fix a train-track representative $\tau = (\Gamma, *, \pi, f)$ and an edge $\widetilde{e}$ of the universal cover $\widetilde{\Gamma}$, from the definition of $\mathcal{Q}_{\widetilde{e}}$, we deduce

Corollary 2.10. *$\mathcal{Q}^2$ and $\mathcal{Q}_{\widetilde{e}}$ are finite-to-one.*

There are finitely many orbits of points in $\widehat{T}_{\Phi^{-1}}$ with strictly more than two pre-images by $\mathcal{Q}^2$ or with stricly more than one pre-image by $\mathcal{Q}_{\widetilde{e}}$. $\square$

From this corollary we can complete Proposition 2.5 by stating that the decomposition obtained there is not a partition (as in Proposition 1.1) but nevertheless intersections are finite.

Proposition 2.11. *Let Φ be an irreducible (with irreducible powers) non-geometric outer automorphism of F_N. Let $T_{\Phi^{-1}}$ be its repelling tree in the boundary of outer space. Let $\tau = (\Gamma, *, \pi, f)$ be a train-track representative for Φ.*

Let $\widetilde{e}$ be an edge of the universal cover $\widetilde{\Gamma}$ lying above the edge e of Γ. Let σ and σ' be two distinct e-paths of length n.

Then the intersection $\Omega_{\widetilde{e},\sigma} \cap \Omega_{\widetilde{e},\sigma'}$ is a finite set.

Proof. By Proposition 1.1, $C_{\widetilde{e},\sigma}$ and $C_{\widetilde{e},\sigma'}$ are disjoint.

Assume by contradiction that there are infinitely many distinct elements $(P_n)_{n\in\mathbb{N}}$ in the intersection. For each n, P_n has at least two pre-images by $\mathcal{Q}_{\widetilde{e}}$ (one starting by σ and one starting by σ'). Applying Corollary 2.10, up to passing to a subsequence, all the points P_n are in the same orbit under the action of F_N: There exist elements $u_n \in F_N$ such that $P_n = u_n P_0$.

From the commutative diagram in Proposition 2.6, for each n, there exists elements $Z_n \in C_{\widetilde{e},\sigma}$ and $Z'_n \in C_{\widetilde{e},\sigma'}$ such that $\mathcal{Q}^2(Z_n) = \mathcal{Q}^2(Z'_n) = P_n$. As $\mathcal{Q}^2$ is equivariant we get

$$\mathcal{Q}^2({u_n}^{-1} Z_n) = \mathcal{Q}^2({u_n}^{-1} Z'_n) = P_0,$$

and as $\mathcal{Q}^2$ is finite-to-one, up to passing to a subsequence we assume that for all n $Z_n = u_n Z_0$ and $Z'_n = u_n Z'_0$.

Again, up to passing to a subsequence we assume that the sequences $(u_n)_{n\in\mathbb{N}}$ and $({u_n}^{-1})_{n\in\mathbb{N}}$ converge to elements U and V respectively in ∂F_N and, as $C_{\widetilde{e},\sigma}$ and $C_{\widetilde{e},\sigma'}$ are compact, that the sequences $(u_n Z_0)_{n\in\mathbb{N}}$ and $(u_n Z'_0)_{n\in\mathbb{N}}$ converge to elements Z and Z'. We also assume that $(u_n V)_{n\in\mathbb{N}}$ converges to an element $W \in \partial F_N$.

The action of F_N on ∂F_N is that of a convergence group, in particular,

$$\forall X \in \partial F_N \smallsetminus \{V\} \quad \lim_{n\to\infty} u_n X = U.$$

As the two ends of Z (resp. Z') are distinct, one of the two ends of Z_0 (resp. Z'_0) is V. Thus Z and Z' are equal to (U, W) or (W, U). As $C_{\widetilde{e},\sigma}$ and $C_{\widetilde{e},\sigma'}$ are disjoint, Z and Z' have the same geometric realisation in reverse order. But $C_{\widetilde{e}}$ does not contain two paths in reverse order. A contradiction. □

We now describe precisely the points with strictly more than one preimage by $\mathcal{Q}_{\widetilde{e}}$. For that we need to assume that the outer automorphism Φ is forward rotationless.

An irreducible (with irreducible powers), non-geometric, outer automorphism $\Phi \in \partial F_N$ is **forward rotationless** (see [FH06]) if for any integer n, for any automorphism ψ in the outer class Φ^n with strictly more than two attracting fixed points in ∂F_N, there exists an automorphism φ in the outer class Φ such that $\varphi^n = \psi$ and such that each fixed point of ψ is a fixed point of φ.

From the following Proposition, we see that this extra hypothesis will not restrict the scope of our results.

Proposition 2.12 ([GJLL98])**.** *There exists a constant K_N depending only on N such that for any irreducible (with irreducible powers), non-geometric outer automorphism Φ, the power Φ^{K_N} is forward rotationless.*

This Proposition is true for any outer automorphism but we restricted ourself to the easier case of irreducible non-geometric automorphisms.

Proposition 2.13. *Let Φ be an irreducible (with irreducible powers) non-geometric forward rotationless outer automorphism of F_N. Let $T_{\Phi^{-1}}$ be its repelling tree in the boundary of outer space. Let $\tau = (\Gamma, *, \pi, f)$ be a train track representative for Φ and let $\widetilde{e}$ be an edge of the universal cover $\widetilde{\Gamma}$ of Γ.*

Let P be a point in $\Omega_{\widetilde{e}}$ with stricly more than one pre-image by $\mathcal{Q}_{\widetilde{e}}$. Then any prefix-suffix representation $\sigma \in \mathcal{Q}_{\widetilde{e}}^{-1}(P)$ is pre-periodic.

Moreover, there exists a homothety H of $T_{\Phi^{-1}}$ associated to an automorphism $\varphi \in \Phi$ and to a lift $\widetilde{f}$ of f such that $H(P) = P$ and for any pre-image $\sigma \in \mathcal{Q}_{\widetilde{e}}^{-1}(P)$ the sequence $(\sigma(\widetilde{e}, \widetilde{f}, n))_{n\in\mathbb{N}}$ is pre-periodic.

Proof. As P has stricly more than one pre-image by $\mathcal{Q}_{\widetilde{e}}$, it has at least three different pre-images by $\mathcal{Q}$. Let $\varphi \in \Phi$ be an automorphism in the outer class Φ and let H be the associated homothety of $T_{\Phi^{-1}}$. For each integer n, by Lemma 2.2, $\mathcal{Q}^{-1}(H^n(P)) = \varphi^n(\mathcal{Q}^{-1}(P))$ and by Proposition 2.9, there exists an integer $n \geq 1$ and an element u of F_N such that $uH^n(P) = P$.

Let σ be a pre-image by $\mathcal{Q}_{\widetilde{e}}$ of P. For any line $Z \in C_{\widetilde{e},\sigma}$, by Lemma 2.3, and for any $k \in \mathbb{N}$, $(u(\partial^2\varphi)^n)^k(Z)$ is also in the $\mathcal{Q}^2$ fiber of P. By Corollary 2.10, $C_{\widetilde{e},\sigma}$ is finite and thus there exists $k \geq 1$ such that

$$\forall Z \in C_{\widetilde{e}}, \mathcal{Q}^2(Z) = P \Rightarrow (u(\partial^2\varphi)^n)^k(Z) = Z.$$

Thus, elements $Z \in C_{\widetilde{e}}$ in the $\mathcal{Q}^2$ fiber of P are fixed points of the automorphism $\psi = (i_u \circ \varphi^n)^k$ of the outer class Φ^{nk}. Moreover as $\mathcal{Q}^2(Z) = P$, they are in the attracting lamination L_Φ and thus there two ends are attracting fixed points of ψ. As Φ was assumed to be forward rotationless, there exists an automorphism φ' in the outer class Φ that fixes all the elements $Z \in C_{\widetilde{e}}$ such that $\mathcal{Q}^2(Z) = P$. Let now H' be the homothety associated to φ', then $H'(P) = P$ and applying Proposition 1.6 we proved the Proposition. $\square$

A point P in $\Omega_{\widetilde{e}}$ is **first-singular**, if there exists two distinct e-paths σ and σ' of length 1 such that $P \in \Omega_{\widetilde{e},\sigma} \cap \Omega_{\widetilde{e},\sigma'}$. As there are finitely many e-path of length 1, from Proposition 2.11 we get that there are finitely many first-singular points in $\Omega_{\widetilde{e}}$.

We prove the following technical result that we will use in the sequel.

Lemma 2.14. *Let $\widetilde{e}$ be an edge of $\widetilde{\Gamma}$ that lies above an edge e of Γ. Let n be an integer. Let N be a set of e-paths of length n such that*

$$\forall \sigma, \sigma' \in N, \Omega_{\widetilde{e},\sigma} \cap \Omega_{\widetilde{e},\sigma'} \neq \emptyset.$$

The cardinality of N is bounded above by a constant C_1 depending only on Φ.

Proof. Let σ_0 be the common prefix of all the elements of N. By self-similarity, we can replace $\Omega_{\widetilde{e}}$ by $\Omega_{\widetilde{e},\sigma_0}$ and N by the set N' of suffixes σ' of elements $\sigma = \sigma_0.\sigma'$ of N. Thus we assume that σ_0 has zero-length and that N has strictly more than 1 element.

For each σ in N, the set $\Omega_{\widetilde{e},\sigma}$ contains a first-singular point P and thus σ is the prefix of length n of one of the finitely many pre-images by $\mathcal{Q}_{\widetilde{e}}$ of P. ☐

2.9 Hausdorff dimension and measure

We refer to the book of K. Falconer [Fal90] for definitions of the Hausdorff dimension and measure.

For a metric space (A, d), for any $\varepsilon > 0$, and $k > 0$, let

$$\mathcal{H}^k_\varepsilon(A) = \inf \sum_{i\in\mathbb{N}} |A_i|^k \in \mathbb{R}^+ \cup \{\infty\}$$

where the infimum is taken over all coverings $(A_i)_{i\in\mathbb{N}}$ of A such that the diameter $|A_i|$ of each A_i is smaller than ε. If A is compact, there are finite coverings of A with closed balls of diameter ε, therefore $\mathcal{H}^k_\varepsilon(A)$ is finite.

For a homothety H of ratio λ, one has

$$\mathcal{H}^k_\varepsilon(H(A)) = \lambda^k \mathcal{H}^k_{\lambda\varepsilon}(A).$$

For a fixed $k > 0$, $\varepsilon \mapsto \mathcal{H}^k_\varepsilon(A)$ is decreasing and the **Hausdorff measure in dimension** k of A is

$$\mathcal{H}^k(A) = \lim_{\varepsilon\to 0} \mathcal{H}^k_\varepsilon(A) = \sup_{\varepsilon>0} \mathcal{H}^k_\varepsilon(A) \in \mathbb{R}^+ \cup \{\infty\}.$$

Again, for a homothety H of ratio λ, one has

$$\mathcal{H}^k(H(A)) = \lambda^k \mathcal{H}^k(A).$$

The map $k \mapsto \mathcal{H}^k(A)$ is decreasing and takes values in $\{0, \infty\}$ except in at most one point. The **Hausdorff dimension** of A is

$$\mathrm{Hdim}(A) = \inf\{k \mid \mathcal{H}^k(A) = 0\} = \sup\{k \mid \mathcal{H}^k(A) = \infty\} \in \mathbb{R}^+ \cup \{\infty\}.$$

From these definitions it is classical to deduce that the Hausdorff dimension of a countable union $\cup_{i\in\mathbb{N}}X_i$ of subspaces of A is the supremum of the dimensions of the X_i. In particular the Hausdorff dimension of the limit set Ω is the maximum of the Hausdorff dimension of the compact subsets $\Omega_{\widetilde{e}}$, for all edges $\widetilde{e}$ of $\widetilde{\Gamma}$.

2.10 Main Theorem

The usual context to compute the Hausdorff dimension of a self-similar set (or of a graph directed construction) is inside $\mathbb{R}^n$ which is not the case here. Also, the classical hypothesis to get the lower bound on the Hausdorff dimension is by using the open set condition, of which we need to use a non-classic version. We refer to [MW88] and [Edg08] for computation of the Hausdorff dimension of graph directed constructions and before them to the original article of J. Hutchinson [Hut81] in the case of an iterated function system.

We are now ready to state and prove our main theorem.

Theorem 2.15. *Let Φ be an irreducible (with irreducible powers) outer automorphism of the free group F_N. Let $\tau = (\Gamma, *, \pi, f)$ be an irreducible train-track representative for Φ and let $T_{\Phi^{-1}}$ be the repelling tree of Φ.*

The limit set $\Omega \subseteq \overline{T}_{\Phi^{-1}}$, and for each edge $\widetilde{e}$ of $\widetilde{\Gamma}$, the set $\Omega_{\widetilde{e}} \subseteq \overline{T}_{\Phi^{-1}}$ have Hausdorff dimension

$$\delta = \mathrm{Hdim}(\Omega) = \frac{\ln \lambda_\Phi}{\ln \lambda_{\Phi^{-1}}}$$

where λ_Φ and $\lambda_{\Phi^{-1}}$ are the expansion factors of Φ and Φ^{-1} respectively.

Proof. The limit set Ω is the union of translates by elements of F_N of any $\Omega_{\widetilde{e}}$. Thus the Hausdorff dimensions of these sets are all equal.

The repelling tree $T_{\Phi^{-1}}$ and the attracting lamination L_Φ do not change if we replace Φ by a power. Also, the expansion factor of a power Φ^n is λ_Φ^n. Thus by Proposition 2.12, up to replacing Φ by a suitable power, we assume that Φ is forward rotationless.

For each edge e of Γ we choose one of its lifts in the universal cover $\widetilde{\Gamma}$ and we denote by Ω_e the corresponding subset of $\overline{T}_{\Phi^{-1}}$. We denote by $E(\Gamma)$ the set of edges of Γ.

From Proposition 2.5, we see that each of the pieces Ω_e is covered by finitely many translates of homothetic copies of the $\Omega_{e'}$. The number of copies used is given by the corresponding row in the transition matrix M

of the train-track τ. From the definition of the Hausdorff dimension it is straightforward to deduce for any $\varepsilon > 0$ and any $k > 0$

$$(\mathcal{H}^k_{\frac{\varepsilon}{\lambda_{\Phi^{-1}}}}(\Omega_e))_{e\in E(\Gamma)} \leq M(\mathcal{H}^k_{\frac{\varepsilon}{\lambda_{\Phi^{-1}}}}(H(\Omega_{e'})))_{e'\in E(\Gamma)} = \frac{1}{(\lambda_{\Phi^{-1}})^k} M(\mathcal{H}^k_{\varepsilon}(\Omega_{e'}))_{e'\in E(\Gamma)}$$

where the comparison between these positive vectors is made coordinatewise.

As the Perron-Frobenius eigen-value of M is the expansion factor λ_Φ of Φ, we get by iteration that if $k > \delta = \frac{\ln \lambda_\Phi}{\ln \lambda_{\Phi^{-1}}}$ for any edge $\widetilde{e}$ of $\widetilde{\Gamma}$, $\mathcal{H}^k(\Omega_{\widetilde{e}}) = 0$. Also, if $k = \delta$ we get that $(\mathcal{H}^\delta(\Omega_e))_{e\in E(\Gamma)}$ is bounded above by the Perron-Frobenius eigen-vector of M. In particular, for any edge $\widetilde{e}$ of Γ, $\mathcal{H}^\delta(\Omega_{\widetilde{e}}) < \infty$.

This gives an upper bound for the Hausdorff dimension of each of the $\Omega_{\widetilde{e}}$ and an upper bound for the Hausdorff measure in dimension δ.

We now proceed to get the lower bound of the Hausdorff dimension and measure. This involves describing quantitatively how much the maps $\mathcal{Q}_{\widetilde{e}}$ fails to be injective and to evaluate how much they contract the distances.

For two subsets C and C' of $\overline{T}_{\Phi^{-1}}$ we denote by $g(C, C')$ the size of the gap between them:

$$g(C, C') = \inf\{d(P, P') \mid P \in C, P' \in C'\}.$$

We decompose the proof into three Lemmas.

Lemma 2.16. *Let σ and σ' be two e-paths of length n in Σ_e such that $\Omega_{\widetilde{e},\sigma} \cap \Omega_{\widetilde{e},\sigma'} = \emptyset$.*

There exists a constant $C_2 > 0$ depending only on Φ such that the gap between $\Omega_{\widetilde{e},\sigma}$ and $\Omega_{\widetilde{e},\sigma'}$ is bigger than $\frac{C_2}{(\lambda_{\Phi^{-1}})^n}$:

$$g(\Omega_{\widetilde{e},\sigma}, \Omega_{\widetilde{e},\sigma'}) > \frac{C_2}{(\lambda_{\Phi^{-1}})^n}.$$

Proof. By self-similarity, up to removing a common prefix to σ and σ' and applying a homothety H, we assume that σ and σ' have different first edges σ_1 and σ'_1.

Let $0 \leq p < n$ be the maximal length of prefixes σ_p and σ'_p of σ and σ' respectively such that $\Omega_{\widetilde{e},\sigma_p} \cap \Omega_{\widetilde{e},\sigma'_p} \neq \emptyset$. As $\Omega_{\widetilde{e},\sigma} \subseteq \Omega_{\widetilde{e},\sigma_{p+1}}$ and $\Omega_{\widetilde{e},\sigma'} \subseteq \Omega_{\widetilde{e},\sigma'_{p+1}}$ we get that

$$g(\Omega_{\widetilde{e},\sigma}, \Omega_{\widetilde{e},\sigma'}) \geq g(\Omega_{\widetilde{e},\sigma_{p+1}}, \Omega_{\widetilde{e},\sigma'_{p+1}}) > 0.$$

Thus, replacing σ and σ' by their prefixes of length $p + 1$, we assume that $p + 1 = n = |\sigma| = |\sigma'|$.

If $n = 1$ (that is to say $p = 0$) then there are only finitely many choices of paths σ and σ' and C_2 has to be smaller than the minimum of the gaps between all such possible choices of $\Omega_{\widetilde{e},\sigma}$ and $\Omega_{\widetilde{e},\sigma'}$.

Thus we assume that $n > 1$ (and that $p = n - 1 > 0$).

Let P be a point in $\Omega_{\widetilde{e},\sigma_{n-1}} \cap \Omega_{\widetilde{e},\sigma'_{n-1}}$. As σ and σ' does not have common prefixes, P is one of the finitely many first-singular points in $\Omega_{\widetilde{e}}$. Let Z and Z' be pre-images of P by $\mathcal{Q}^2$ in $C_{\widetilde{e},\sigma_{n-1}}$ and $C_{\widetilde{e},\sigma'_{n-1}}$ respectively. The point P has at least two different pre-images by $\mathcal{Q}_{\widetilde{e}}$, thus we can use Proposition 2.13 to get that the pre-images by $\mathcal{Q}_{\widetilde{e}}$ of Z and Z' are pre-periodic. σ_{n-1} and σ'_{n-1} are prefixes of two of these pre-images. We also get a homothety H of $T_{\Phi^{-1}}$ and an associated automorphisms $\varphi \in \Phi$ and lift $\widetilde{f}$ of f such that the sequences $(\sigma_{n-1}(\widetilde{e}, \widetilde{f}, k))_{0\leq k\leq n-1}$ and $(\sigma'_{n-1}(\widetilde{e}, \widetilde{f}, k))_{0\leq k\leq n-1}$ only takes finitely many values. As the prefix-suffix automaton Σ is finite the terminal edges $\widetilde{e}_n = \sigma(\widetilde{e}, \widetilde{f}, n)$ and $\widetilde{e}'_n = \sigma'(\widetilde{e}, \widetilde{f}, n)$ takes only finitely possible values.

From our definitions

$$\Omega_{\widetilde{e},\sigma} = H^n(\Omega_{\widetilde{e}_n}) \text{ and } \Omega_{\widetilde{e},\sigma'} = H^n(\Omega_{\widetilde{e}'_n}),$$

thus, the lower bound of the gap is now given by:

$$g(\Omega_{\widetilde{e},\sigma}, \Omega_{\widetilde{e},\sigma'}) = \frac{1}{(\lambda_{\Phi^{-1}})^n} g(\Omega_{\widetilde{e}_n}, \Omega_{\widetilde{e}'_n}).$$

The existence of the constant C_2 follows from the finiteness of the number of possible choices for $\widetilde{e}_n$ and $\widetilde{e}'_n$.

For the sake of clarity, let us review this finiteness again. Using the action of F_N by isometries, we only need to consider one choice of a lift $\widetilde{e}$ of each edge e of Γ. For each of these $\widetilde{e}$ we consider the finitely many first-singular points P in $\Omega_{\widetilde{e}}$. For each of these first-singular points P we consider the associated lift $\widetilde{f}$ of f as given by Proposition 2.13 and their finitely many pre-periodic pre-images σ by $\mathcal{Q}_{\widetilde{e}}$. Proposition 2.13 states that the sequence $(\sigma(\widetilde{e}, \widetilde{f}, k))_{k\in\mathbb{N}}$ is contained in a finite set $E_{\widetilde{e},P}$ of edges of $\widetilde{\Gamma}$. Finally the edges $\widetilde{e}_n$ and $\widetilde{e}'_n$ are among the finitely many edges of $\widetilde{\Gamma}$ such that $\widetilde{f}(\widetilde{e}_n)$ and $\widetilde{f}(\widetilde{e}'_n)$ contain one of the edges of $E_{\widetilde{e},P}$. □

Let P be a point in $\Omega_{\widetilde{e}}$ and let n be an integer. We consider the following subset of Σ_e:

$$N(P, n) = \{\sigma \in \Sigma_e \mid |\sigma| = n, g(P, \Omega_{\widetilde{e},\sigma}) \leq (\frac{1}{\lambda_{\Phi^{-1}}})^n\}.$$

The use of the set $N(P,n)$ is classical while proving lower bounds for Hausdorff dimension and measure.

Lemma 2.17. *There exists a constant C_3 depending only on the outer automorphism Φ such that for any point P in $\Omega_{\widetilde{e}}$ and any integer n, the set $N(P,n)$ has at most C_3 elements.*

Proof. Let $k = \lceil \frac{\ln \frac{2}{C_2}}{\ln \lambda_{\Phi^{-1}}} \rceil$. For any elements σ and σ' in $N(P,n)$, by definition $g(\Omega_{\widetilde{e},\sigma}, \Omega_{\widetilde{e},\sigma'}) \leq \frac{2}{\lambda_{\Phi^{-1}}{}^n}$. We consider the prefixes, σ_{n-k} and σ'_{n-k} of σ and σ' of length $n-k$. The sets $\Omega_{\widetilde{e},\sigma_{n-k}}$ and $\Omega_{\widetilde{e},\sigma'_{n-k}}$ contains the sets $\Omega_{\widetilde{e},\sigma}$ and $\Omega_{\widetilde{e},\sigma'}$, thus

$$g(\Omega_{\widetilde{e},\sigma_{n-k}}, \Omega_{\widetilde{e},\sigma'_{n-k}}) \leq g(\Omega_{\widetilde{e},\sigma}, \Omega_{\widetilde{e},\sigma'}) \leq \frac{2}{(\lambda_{\Phi^{-1}})^n} \leq \frac{C_2}{(\lambda_{\Phi^{-1}})^{n-k}}.$$

From Lemma 2.16 we get that these two sets are not disjoints:

$$\Omega_{\widetilde{e},\sigma_{n-k}} \cap \Omega_{\widetilde{e},\sigma'_{n-k}} \neq \emptyset.$$

From Lemma 2.14 the number of prefixes of length $n-k$ of $N(P,n)$ is bounded above by C_1. Thus the cardinality of $N(P,n)$ is bounded above by $C_3 = C_1.(E)^k$, where E is the number of edges of the prefix-suffix automaton Σ. □

For a point P in $\Omega_{\widetilde{e}}$ and $r > 0$ we denote by $B(P,r)$ the ball of radius r in $\Omega_{\widetilde{e}}$.

Lemma 2.18. *There exists a constant C_4 depending only on Φ such that*

$$\nu_\Phi(B(P,r)) \leq C_4 r^\delta.$$

Proof. Let $n = \lfloor \frac{\ln \frac{1}{r}}{\ln \lambda_{\Phi^{-1}}} \rfloor$. For any point Q in $B(P,r)$, let σ be the prefix of length n of some pre-image by $\mathcal{Q}_{\widetilde{e}}$ of Q. Then $Q \in \Omega_{\widetilde{e},\sigma}$ and

$$g(P, \Omega_{\widetilde{e},\sigma}) \leq d(P,Q) \leq r \leq (\frac{1}{\lambda_{\Phi^{-1}}})^n.$$

Thus we have proved that

$$B(P,r) \subseteq \bigcup_{\sigma \in N(P,n)} \Omega_{\widetilde{e},\sigma}.$$

Applying the push-forward ν_Φ of the attracting current μ_Φ we get

$$\nu_\Phi(B(P,r)) \leq \sum_{\sigma \in N(P,n)} \nu_\Phi(\Omega_{\widetilde{e},\sigma})$$

For each $\sigma \in N(P,n)$

$$\nu_\Phi(\Omega_{\widetilde{e},\sigma}) = \frac{\nu_\Phi(\Omega_{\sigma(\widetilde{e},\widetilde{f},n)})}{{\lambda_\Phi}^n} \leq r^\delta\, \nu_\Phi(\Omega_{\sigma(\widetilde{e},\widetilde{f},n)}),$$

and thus

$$\nu_\Phi(B(P,r)) \leq C_3\, r^\delta \max\{\nu_\Phi(\Omega_{\widetilde{e}'}) \mid \widetilde{e}'\}$$

which proves the Lemma. □

From Lemma 2.18 we deduce that the Hausdorff measure in dimension δ is bounded below by the push forward of the attracting current μ_Φ. This proves that the Hausdorff dimension of Ω is bounded below by δ. □

From the above proof we get that the Hausdorff measure in dimension δ on the limit set Ω is not constant. Pulling back this measure to the attracting lamination by the map $\mathcal{Q}^2$ we get another current supported by the attracting lamination L_Φ. But we know that the attracting lamination is uniquely ergodic thus we have proved:

Theorem 2.19. *The pushforward ν_Φ of the attracting current μ_Φ to the limit set Ω is equal to the Hausdorff measure in dimension δ.* □

Once again this equality is to be understood up to a multiplicative constant.

2.11 Compact heart of trees

In this section we relate the sets Ω and $\Omega_{\widetilde{e}}$ of the previous section to the compact heart of $T_{\Phi^{-1}}$ as defined in [CHL09].

We fix a basis $\mathcal{A}$ of F_N. This is equivalent to fixing a chart $(R_\mathcal{A}, *, \pi)$ where $R_\mathcal{A}$ is the rose with N petals and π the corresponding marking isomorphism. Elements ∂F_N are identified with infinite reduced words in $\mathcal{A}^{\pm 1}$.

The unit-cylinder $C_\mathcal{A}(1)$ of $\partial^2 F_N$ is the set of lines that goes through the origin, or equivalently, it is the set of pairs of infinite reduced words (X,Y) with distinct first letters. This is a compact subset of $\partial^2 F_N$ whose translates cover the double boundary: $F_N.C_\mathcal{A}(1) = \partial^2 F_N$.

For a basis $\mathcal{A}$ of F_N the **compact limit set** of the repelling tree $T_{\Phi^{-1}}$ is defined in [CHL09] by

$$\Omega_\mathcal{A} = \mathcal{Q}^2(L(T_{\Phi^{-1}}) \cap C_\mathcal{A}(1)).$$

It is a compact subset of $\overline{T}_{\Phi^{-1}}$.

From [CHK⁺08] we know that the dual lamination $L(T_{\Phi^{-1}})$ is the diagonal closure of the attracting lamination L_Φ. In particular

$$\Omega = \mathcal{Q}^2(L_\Phi) = \mathcal{Q}^2(L(T_{\Phi^{-1}}))$$

$$\Omega_{\mathcal{A}} = \mathcal{Q}^2(L(T_{\Phi^{-1}}) \cap C_{\mathcal{A}}(1)) = \mathcal{Q}^2(L_\Phi \cap C_{\mathcal{A}}(1)).$$

The translates of the compact limit set $\Omega_{\mathcal{A}}$ cover the limit set Ω: $F_N.\Omega_{\mathcal{A}} = \Omega$.

As the Hausdorff dimension does not increase by taking countable unions we get that the Hausdorff dimension of $\Omega_{\mathcal{A}}$ is δ.

The compact heart, $K_{\mathcal{A}}$ of $T_{\Phi^{-1}}$ is the convex hull $\Omega_{\mathcal{A}}$. Recall that the tree $T_{\Phi^{-1}}$ can be covered by countably many intervals (thus $T_{\Phi^{-1}}$ has Hausdorff dimension 1), and note that this is not the case of its metric completion $\overline{T}_{\Phi^{-1}}$. The compact heart $K_{\mathcal{A}}$ is a subset of the union $\Omega_{\mathcal{A}} \cup T_{\Phi^{-1}}$. We get

Theorem 2.20. *Let Φ be an irreducible (with irreducible powers) outer automorphism of F_N. Let $\mathcal{A}$ be a basis of F_N. Let $T_{\Phi^{-1}}$ be the repelling tree of Φ. Let $\Omega_{\mathcal{A}}$ be the compact limit set and $K_{\mathcal{A}}$ be the compact heart of $T_{\Phi^{-1}}$ with respect to $\mathcal{A}$. Then*

$$\mathrm{Hdim}(\Omega_{\mathcal{A}}) = \delta = \frac{\ln \lambda_\Phi}{\ln \lambda_{\Phi^{-1}}} \text{ and } \mathrm{Hdim}(K_{\mathcal{A}}) = \max\{1, \frac{\ln \lambda_\Phi}{\ln \lambda_{\Phi^{-1}}}\}. \quad \square$$

3 Examples

In this section, we illustrate our result with two examples of irreducible (with irreducible powers) automorphisms.

3.1 Boshernitzan-Kornfeld example

In [BK95] the following automorphism of F_3 is studied:

$$\begin{array}{rlcl} \varphi: & a & \mapsto & b \\ & b & \mapsto & caaa \\ & c & \mapsto & caa \end{array}$$

Let Φ be its outer class. We regard φ as a homeomorphism of the rose with 3 petals to get a train-track representative of Φ and the corresponding prefix-suffix automaton Σ, see Figure 1.

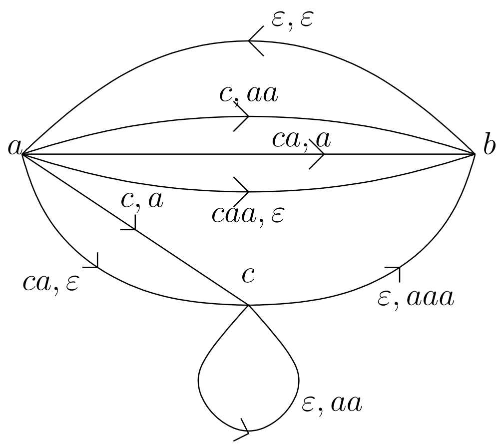

Figure 1: Prefix-suffix automaton Σ for Boshernitzan-Kornfeld automorphism

The transition matrix M_Φ and the expansion factor λ_Φ (which is the Perron-Frobenius eigen-value of M_Φ) are

$$M_\Phi = \begin{pmatrix} 0 & 3 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } \lambda_\Phi \approx 2.170.$$

The inverse automorphism

$$\begin{array}{rccl} \varphi^{-1}: & a & \mapsto & c^{-1}b \\ & b & \mapsto & a \\ & c & \mapsto & cb^{-1}cb^{-1}c \end{array}$$

also defines on the rose with 3 petals a train-track representative of Φ^{-1}. The transition matrix and the expansion factor of Φ^{-1} are

$$M_{\Phi^{-1}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 3 \end{pmatrix} \text{ and } \lambda_{\Phi^{-1}} \approx 3.214.$$

(note that positive and negative letters are both counted as one in the transition matrix).

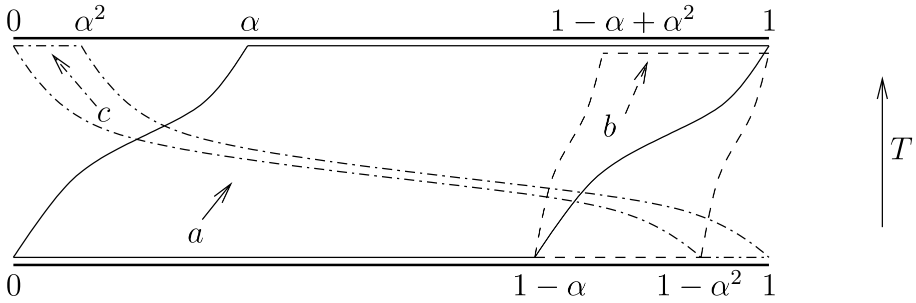

Figure 2: Interval translation of Boshernitzan-Kornfeld

M. Boshernitzan and I. Kornfeld associate this automorphism to an interval translation map.

Let $I = [0; 1]$ be the unit interval and let $\alpha = \frac{1}{\lambda_{\Phi^{-1}}} \approx 0.311$ be the positive root of $\alpha^3 - \alpha^2 - 3\alpha + 1 = 0$. The piecewise translation $T : [0; 1] \to [0; 1]$ restricts to a translation on each of the three intervals $I_a = [0; 1-\alpha]$, $I_b = [1-\alpha; 1-\alpha^2]$ and $I_c = [1-\alpha^2; 1]$ of translation vector α, α^2 and $\alpha^2 - 1$ respectively.

The fact that φ and T are associated can be seen by looking at the first return map on the interval $[1-\alpha; 1]$.

The repelling tree $T_{\Phi^{-1}}$ of Φ is the tree dual to (the lift to the universal cover of) the vertical foliation of the mapping torus of this interval translation. See [GL95] and [CHL09] for a precise construction of $T_{\Phi^{-1}}$ starting from the interval translation T.

The compact heart $K_{\mathcal{A}}$ of $T_{\Phi^{-1}}$ is the interval I. The restriction of the action of the elements a, b and c of the basis of F_3 to this interval $K_{\mathcal{A}} = I$ are exactly the piecewise exchange of the interval translation map, T. The compact limit set $\Omega_{\mathcal{A}}$ is the limit set of the interval translation map:

$$\Omega_{\mathcal{A}} = \bigcap_{n \geq 0} T^n(I).$$

This is a Cantor set with Hausdorff dimension $\frac{\ln \lambda_\Phi}{\ln \lambda_{\Phi^{-1}}} \approx 0,664$.

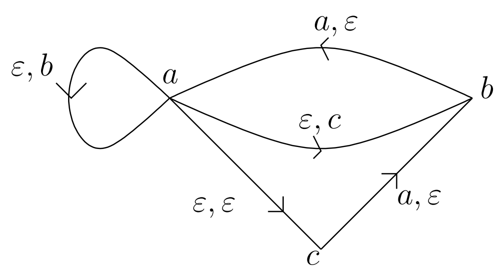

Figure 3: Prefix-suffix automaton Σ for Tribonacci automorphism

3.2 Tribonacci example

The following Tribonacci automorphism of F_3 has long been studied

$$\varphi : \begin{array}{rcl} a & \mapsto & ab \\ b & \mapsto & ac \\ c & \mapsto & a \end{array}$$

It is associated to what is known as *the* Rauzy fractal, and X. Bressaud studied its repelling tree (see [Bre07]).

Let Φ be its outer class. We regard φ as a homeomorphism of the rose with 3 petals to get a train-track representative of Φ and the associated prefix-suffix automaton Σ, see Figure 3.

The transition matrix and the expansion factor are

$$M_\Phi = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \lambda_\Phi \approx 1.839.$$

A train-track representative of Φ^{-1} is given by the graph Γ of Figure 4, the homeomorphism f and the marking π:

$$f : \begin{array}{rcl} A & \mapsto & DC \\ B & \mapsto & D^{-1}A \\ C & \mapsto & B \\ D & \mapsto & C^{-1} \end{array} \qquad \pi : \begin{array}{rcl} a & \mapsto & A \\ b & \mapsto & DB \\ c & \mapsto & DC \end{array}$$

The prefix-suffix automaton is given in Figure 5.

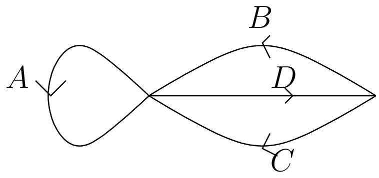

Figure 4: Graph Γ of a train-track representative of the inverse of Tribonacci automorphism

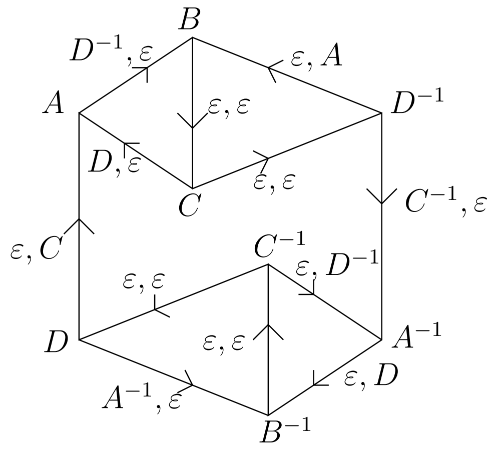

Figure 5: Prefix-suffix automaton Σ for the inverse of tribonacci automorphism

The outer automorphism Φ^{-1} has transition matrix and expansion factor

$$M_{\Phi^{-1}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \text{ and } \lambda_{\Phi^{-1}} \approx 1.395.$$

The repelling tree $T_{\Phi^{-1}}$ has a connected limit set $\Omega_{\mathcal{A}} = K_{\mathcal{A}}$ which is of Hausdorff dimension $\delta = \frac{\ln \lambda_\Phi}{\ln \lambda_{\Phi^{-1}}} \approx 1.829$. The $\mathbb{R}$-tree $K_{\mathcal{A}}$, although compact, has Hausdorff dimension strictly bigger than 1.

X. Bressaud has drawn nice pictures of (approximations of) the fractal tree $K_{\mathcal{A}}$ inside the Rauzy fractal (see [BC07]).

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